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ABSTRACT

The papers in this document follow the order of the meeting and consist of two guest lecturers and reports from four working groups; two topic group presentations are noted but not included. One lecture, delivered by Peter Hilton, discusses the nature of mathematics today and implications for mathematics teaching, while, in the other paper, Stephen I. Brown explores the nature of problem generation in the mathematics curriculum. Working group reports concern statistical thinking, training in diagnosis and remediation for teachers, mathematics and language, and the influence of computer science on the undergraduate mathematics curriculum. Topic groups heard presentations by Daniel Kahneman on intuitions and fallacies in reasoning about probability and by Tom Kieren on mathematics curriculum development in Canada. A list of participants is given. (MNS)

GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

PROCEEDINGS OF THE 1983 ANNUAL MEETING

UNIVERSITY OF BRITISH COLUMBIA

VANCOUVER, BRITISH COLUMBIA

JUNE 8-12, 1983

EDITED

BY

CHARLES VERHILLE

CMESG/GCEDM

NOVEMBER 1983

CMESG/GCEDM
1983 Meeting
PROCEEDINGS

Canadian Mathematics Education Study Group/
Groupe canadien d'étude en didactique des mathématiques

1983 Meeting

The seventh annual meeting of the Study Group was held on the UBC campus from June 8th to 12th. About 45 mathematicians and mathematics educators met in working groups and plenary sessions, following a pattern established in earlier meetings. The Study Group continues to fill a unique role, in at least two respects: it brings mathematicians and mathematics educators together where they can meet on equal terms, and it allows time for topics to be followed through, both during each meeting and from one meeting to the next.

The principal guests this year were Peter Hilton (Suny at Binghamton) and Stephen Brown (SUNY at Buffalo). The former spoke on "The nature of mathematics today and implications for mathematics teaching", and the latter on "The nature of problem generation and the mathematics curriculum". Their lectures were stimulating and provocative, but perhaps even more significant were their individual contributions to other parts of the programme and their public dialogues with each other. Scheduled lectures were given by Daniel Kahneman (UBC) on "Intuitions and fallacies in reasoning about probability" and Thomas Kieren (Alberta) on "Mathematics curriculum development in Canada: a projection for the future", and ad hoc presentations were offered by Peter Taylor (Queen's) on "Mathematics as poetry" and John Barry (Manitoba) on "Cross-cultural aspects of teaching mathematics".

This year's working groups took as their subjects "Developing statistical thinking", "Training in diagnosis and remediation for teachers", "Mathematics and language" and "The influence of computer science on the undergraduate mathematics curriculum". At least three of the four groups plan to produce papers or short monographs on the basis of their discussions. An afternoon was set aside for demonstrations of computer software.

There were slightly fewer participants this year than at the last three meetings, perhaps because travelling expense support is less easy to come by. Although meetings would lose their character if the Study Group became too large, it seems a pity that many mathematics departments and faculties of education in Canada were not represented at all. Anyone wanting information about the Study Group may get in touch with the writer or with B.R. Hodgson, Département de mathématiques, Université Laval, Québec, Qué., G1K 7P4.

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EDITOR'S FORWARD

The 1983 CMESG/GCEDM meeting followed the same format used for several years. The agenda included two lectures presented by prominent persons; four working groups, each focusing on a new or continuing theme from previous conferences; two topic groups and continuing groups. In addition, the program included a computer workshop and opportunities for ad hoc sessions.

The lectures were presented by Peter Hilton of the State University of New York at Binghamton and Stephen Brown of the State University of New York at Buffalo. The papers from both lectures are included in these proceedings in their entirety.

Reports from each of the working groups are included.

The text of the Topic Group presented by Daniel Kohneman entitled "Intentions and fallacies in reasoning about probability" is not included. However, some references which include related tectural material are included.

Charles Verhille
EDITOR

LECTURE 1

CURRENT TRENDS IN MATHEMATICS AND FUTURE TRENDS IN MATHEMATICS EDUCATION

BY

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CURRENT TRENDS IN MATHEMATICS AND FUTURE TRENDS IN MATHEMATICS EDUCATION

by

Peter Hilton

1. Introduction

My intention in this talk is to study, grosso modo, the dominant trends in present-day mathematics, and to draw from this study principles that should govern the choice of content and style in the teaching of mathematics at the secondary and elementary levels. Some of these principles will be time-independent, in the sense that they should always have been applied to the teaching of mathematics; others will be of special application to the needs of today's, and tomorrow's, students and will be, in that sense, new. The principles will be illustrated by examples in order to avoid the sort of frustrating vagueness which often accompanies even the most respectable recommendations (thus, 'problem solving [should] be the focus of school mathematics in the 1980's' [1]).

However, before embarking on a talk intended as a contribution to the discussion of how to achieve a successful mathematical education, it would be as well to make plain what are our criteria of success. Indeed, it would be as well to be clear what we understand by successful education, since we would then be able to derive the indicated criteria by specialization.

Let us begin by agreeing that a successful education is one which conduces to a successful life. However, there is a popular, persistent and paltry view of the successful life which we must immediately repudiate. This is the view that success in life is measured by affluence and is manifested by power and influence over others. It is very relevant to my theme to recall that, when Queen Elizabeth was recently the guest of President and Mrs. Reagan in California, the 'successes' who were gathered together to greet her were not Nobel prize-winners, of which California may boast remarkably many, but stars of screen and television. As the London Times described the occasion, "Queen dines with celluloid royalty". It was apparently assumed that the company of Frank Sinatra, embodying the concept of success against which I am inveighing, would be obviously preferable to that of, say, Linus Pauling.

The Reaganist-Sinatrist view of success contributes a real threat to the integrity of education; for education should certainly never be expected to conduce to that kind of success. At worst, this view leads to a complete distortion of the educational process; at the very least, it allies education far too closely to specific career objectives, an alliance which unfortunately has the support of many parents naturally anxious for their children's success.

We would replace the view we are rejecting by one which emphasizes the kind of activity in which an individual indulges, and the motivation for so indulging, rather than his, or her, accomplishment in that activity. The realization of the individual's potential is surely a mark of success in life. Contrasting our view with that which we are attacking, we should seek power over ourselves, not over other people; we should seek the knowledge and understanding to give us power and control over things, not people. We should want to be rich but in spiritual rather than material resources. We should want to influence people, but by the persuasive force of our argument and example, and not by the pressure we can exert by our control of their lives and, even more sinisterly, of their thoughts.

It is absolutely obvious that education can, and should, lead to a successful life, so defined. Moreover, mathematical education is a particularly significant component of such an education. This is true for two reasons. On the one hand, I would state dogmatically that mathematics is one of the human activities, like art, literature, music, or the making of good shoes, which is intrinsically worthwhile. On the other hand, mathematics is a key element in science and technology and thus vital to the understanding, control and development of the resources of the world around us. These two aspects of mathematics, often referred to as pure mathematics and applied mathematics, should both be present in a well-balanced, successful mathematics education.

Let me end these introductory remarks by referring to a particular aspect of the understanding and control to which mathematics can contribute so much. Through our education we hope to gain knowledge. We can only be said to really know something if we know that we know it. A sound education should enable us to distinguish between what we know and what we do not know; and it is a deplorable fact that so many people today, including large numbers of pseudosuccesses but also, let us admit, many members of our own academic community, seem not to be able to make the distinction. It is of the essence of genuine mathematical education that it leads to understanding and skill; short cuts to the acquisition of skill, without understanding, are often favored by self-confident pundits of mathematical education, and the results of taking such short cuts are singularly unfortunate for the young traveller. The victims, even if 'successful', are left precisely in the position of not knowing mathematics and not knowing they know no mathematics. For most, however, the skill evaporates or, if it does not, it becomes out-dated. No real ability to apply quantitative reasoning to a changing world has been learned, and the most frequent and natural result is the behaviour pattern known as 'mathematics avoidance'. Thus does it transpire that so many prominent citizens exhibit both mathematics avoidance and unawareness of ignorance.

This then is my case for the vital role of a sound mathematical education, and from these speculations I derive my criteria of success.

2. Trends in Mathematics Today

The three principal broad trends in mathematics today I would characterize as (i) variety of applications, (ii) a new unity in the mathematical sciences, and (iii) the ubiquitous presence of the computer. Of course, these are not independent phenomena, indeed they are strongly interrelated, but it is easier to discuss them individually.

The increased variety of application shows itself in two ways. On the one hand, areas of science, hitherto remote from or even immune to mathematics, have become 'infected'. This is conspicuously true of the social sciences, but is also a feature of present-day theoretical biology. It is noteworthy that it is not only statistics and probability which are now applied to the social sciences and biology; we are seeing the application of fairly sophisticated areas of real analysis, linear algebra and combinatorics, to name but three parts of mathematics involved in this process.

But another contributing factor to the increased variety of applications is the conspicuous fact that areas of mathematics, hitherto regarded as impregnably pure, are now being applied. Algebraic geometry is being applied to control theory and the study of large-scale systems; combinatorics and graph theory are applied to economics; the theory of fibre bundles is applied to physics; algebraic invariant theory is applied to the study of error correcting codes. Thus the distinction between pure and applied mathematics is seen now not to be based on content but on the attitude and motivation of the mathematician. No longer can it be argued that certain mathematical topics can safely be neglected by the student contemplating a career applying mathematics. I would go further and argue that there should not be a sharp distinction between the methods of pure and applied mathematics. Certainly such a distinction should not consist of a greater attention to rigour in the pure community; for the applied mathematician needs to understand very well the domain of validity of the methods being employed, and to be able to analyse how stable the results are and the extent to which the methods may be modified to suit new situations.

These last points gain further significance if one looks more carefully at what one means by 'applying mathematics'. Nobody would seriously suggest that a piece of mathematics be stigmatized as inapplicable just because it happens not yet to have been applied. Thus a fairer distinction than that between 'pure' and 'applied' mathematics would seem to be one between 'inapplicable' and 'applicable' mathematics, and our earlier remarks suggest we should take the experimental view that the intersection of inapplicable mathematics and good mathematics is probably empty. However, this view comes close to being a subjective certainty if one understands that applying mathematics is very often not a single-stage process. We wish to study a 'real world' problem; we form a scientific model of the problem and then construct a mathematical model to reason about the scientific or conceptual model (see [2]). However, to reason <u>within</u> the mathematical model, we may well feel compelled to construct a new mathematical model which embeds our original model in a more abstract conceptual context; for example, we may study a particular partial differential equation by bringing to bear a general theory of elliptic differential operators. Now the process of modeling a mathematical situation is a 'purely' mathematical process, but it is apparently not confined to pure mathematics! Indeed, it may well be empirically true that it is more often found in the study of applied problems than in research in pure mathematics. Thus we see, first, that the concept of applicable mathematics needs to be broad enough to include parts of mathematics applicable to some area of mathematics which has already been applied; and, second, that the methods of pure and applied mathematics have much more in common than would be supposed by anyone listening to some of their more vociferous advocates. For our purposes now, the lessons for mathematics education to be drawn from looking at this trend in mathematics are twofold; first, the distinction between pure and applied mathematics should not be emphasized in the teaching of mathematics, and, second, opportunities to present applications should be taken wherever appropriate within the mathematics curriculum.

The second trend we have identified is that of a new unification of mathematics. This is discussed at some length in [3], so we will not go into great detail here. We would only wish to add to the discussion in [3] the remark that this new unification is clearly discernible within mathematical research itself. Up to ten years ago the most characteristic feature of this research was the 'vertical' development of autonomous disciplines, some of which were of very recent origin. Thus the community of mathematicians was partitioned into subcommunities united by a common and rather exclusive interest in a fairly narrow area of mathematics (algebraic geometry, algebraic topology, homological algebra, category theory, commutative ring theory, real analysis, complex analysis, summability theory, set theory, etc., etc.). Indeed, some would argue that no real community of mathematicians existed, since specialists in distinct fields were barely able to communicate with each other. I do not impute any fault to the system which prevailed in this period of remarkably vigorous mathematical growth--indeed, I believe it was historically inevitable and thus 'correct'--but it does appear that these autonomous disciplines are now being linked together in such a way that mathematics is being reunified. We may think of this development as 'horizontal', as opposed to 'vertical' growth. Examples are the use of commutative ring theory in combinatorics, the use of cohomology theory in abstract algebra, algebraic geometry, functional analysis and partial differential equations, and the use of Lie group theory in many mathematical disciplines, in relativity theory and in invariant gauge theory.

I believe that the appropriate education of a contemporary mathematician must be broad as well as deep, and that the lesson to be drawn from the trend toward a new unification of mathematics must involve a similar principle. We may so formulate it: we must break down artificial barriers between mathematical topics throughout the student's mathematical education.

The third trend to which I have drawn attention is that of the general availability of the computer and its role in actually changing the face of mathematics. The computer may eventually take over our lives; this would be a disaster. Let us assume this disaster can be avoided; in fact, let us assume further, for the purposes of this discussion at any rate, that the computer plays an entirely constructive role in our lives and in the evolution of our mathematics. What will then be the effects?

The computer is changing mathematics by bringing certain topics into greater prominence--it is even causing mathematicians to create new areas of mathematics (the theory of computational complexity, the theory of automata, mathematical cryptology). At the same time it is relieving us of certain tedious aspects of traditional mathematical activity which it executes faster and more accurately than we can. It makes it possible rapidly and painlessly to carry out numerical work, so that we may accompany our analysis of a given problem with the actual calculation of numerical examples. However, when we use the computer, we must be aware of certain risks to the validity of the solution obtained due to such features as structural instability and round-off error. The computer is especially adept at solving problems involving iterated procedures, so that the method of successive approximations (iteration theory) takes on a new prominence. On the other hand, the computer renders obsolete certain mathematical techniques which have hitherto been prominent in the curriculum--a sufficient example is furnished by the study of techniques of integration.

There is a great debate raging as to the impact which the computer should have on the curriculum (see, for example, [6]). Without taking sides in this debate, it is plain that there should be a noticeable impact, and that every topic must be examined to determine its likely usefulness in a computer age. It is also plain that no curriculum today can be regarded as complete unless it prepares the student to use the computer and to understand its mode of operation. We should include in this understanding a realization of its scope and its limitations; and we should abandon the fatuous idea, today so prevalent in educational theory and practice, that the principal purpose of mathematical education is to enable the child to become an effective computer even if deprived of all mechanical aids!

Let me elaborate this point with the following table of comparisons. On the left I list human attributes and on the right I list the contrasting attributes of a computer when used as a calculating engine. I stress this point because I must emphasize that I am not here thinking of the computer as a research tool in the study of artificial intelligence. I should also add that I am talking of contemporary human beings and contemporary computers. Computers evolve very much faster than human beings so that their characteristics may well undergo dramatic change in the span of a human lifetime. With these caveats, let me display the table.

Humans	Computers
Compute slowly and inaccurately.	Compute fast and accurately.
Get distracted.	Are remorseless, relentless and dedicated.
Are interested in many things at the same time.	Always concentrate and cannot be diverted.
Sometimes give up.	Are incurably stubborn.
Are often intelligent and understanding.	Are usually pedantic and rather stupid.
Have ideas and imagination, make inspired guesses, think.	Can execute 'IF...ELSE' instructions.

Human and Computer Attributes

It is an irony that we seem to teach mathematics as if our objective were to replace each human attribute in the child by the corresponding computer attribute--and this is a society nominally dedicated to the development of each human being's individual capacities. Let us agree to leave to the computer what the computer does best and to design the teaching of mathematics as a generally human activity. This apparently obvious principle has remarkably significant consequences for the design of the curriculum, the topic to which we now turn.

3. <u>The Secondary Curriculum</u>

Let us organize this discussion around the "In and Out' principle. That is, we will list the topics which should be 'In' or strongly emphasized, and the topics which should be 'Out' or very much underplayed. We will also be concerned to recommend or castigate, as the case may be, certain teaching strategies and styles. We do not claim that all our recommendations are strictly contemporary, in the sense that they are responses to the current prevailing changes in mathematics and its uses; some, in perticular those devoted to questions of teaching practice, are of a lasting nature and should, in my judgment, have been adopted long since.

We will present a list of 'In' and 'Out' items, followed by commentary. We begin with the 'Out' category, since this is more likely to claim general attention; and within the 'Out' category we first consider pedagogical techniques.

Out (Secondary Level)

1. Teaching Strategies

Authoritarianism.

Orthodoxy.

Pointlessness.

Pie-in-the-sky motivation.

2. Topics

Tedious hand cacluations.

Complicated trigonometry.

Learning geometrical proofs.

Artificial 'simplifications'.

Logarithms as calculating devices.

Commentary

There should be no need to say anything further about the evils of authoritarianism and pointlessness in presenting mathematics. They disfigure so many teaching situations and are responsible for the common negative attitudes towards mathematics which regard it as unpleasant and useless. By orthodoxy we intend the magisterial attitude which regards one 'answer' as correct and all others as (equally) wrong. Such an attitude has been particularly harmful in the teaching of geometry. Instead of being a wonderful source of ideas and of questions, geometry must appear to the student required to set down a proof according to rigid and immutable rules as a strange sort of theology, with prescribed responses to virtually meaningless propositions.

Pointlessness means unmotivated mathematical process. By 'pie-in-the-sky' motivation we refer to a form of pseudomotivation in which the student is assured that, at some unspecified future date, it will become clear why the current piece of mathematics warrents learning. Thus we find much algebra done because it will be useful in the future in studying the differential and integral calculus--just as much strange arithmetic done at the elementary level can only be justified by the student's subsequent exposure to algebra. One might perhaps also include here the habit of presenting to the student applications of the mathematics being learnt which could only interest the student at a later level of maturity; obviously, if an application is to motivate a student's study of a mathematical topic, the application must be interesting.

With regard to the expendable topics, tedious hand calculations have obviously been rendered obsolete by the availability of hand-calculators and minicomputers. To retain these appalling travesties of mathematics in the curriculum can be explained only by inertia or sadism on the part of the teacher and curriculum planner. It is important to retain the trigonometric functions (especially as functions of real variables) and their basic identities, but complicated identities should be eliminated and tedious calculations reduced to a minimum. Understanding geometric proofs is very important; inventing one's own is a splendid experience for the student; but memorizing proofs is a suitable occupation only for one contemplating a monastic life of extreme asceticism. Much time is currently taken up with the student processing a mathematical expression which came from nowhere, involving a combination of parentheses, negatives, and fractions, and reducing the expression to one more socially acceptable. This is absurd; but, of course, the student must learn how to substitute numerical values for the variables appearing in a natural mathematical expression.

Let us now turn to the positive side. Since, as our first recommendation below indicates, we are proposing an integrated approach to the curriculum, the topics we list are rather of the form of modules than full-blown courses.

<u>In</u> (Secondary Level)

1. <u>Teaching Strategies</u>

   An integrated approach to the curriculum, stressing the interdependence of the various parts of mathematics.

   Simple applications.

   Historical references.

   Flexibility.

   Exploitation of computing availability.

2. <u>Topics</u>

   Geometry and Algebra (e.g., linear and quadratic functions, equations and inequalities).

   Probability and statistics.

   Approximation and estimation, scientific notation.

   Iterative procedures, successive approximation.

   Rational numbers, ratios and rates.

   Arithmetic mean and geometric mean (and harmonic mean).

   Elementary number theory.

   Paradoxes.

Commentary

With respect to teaching strategies, our most significant recommendation is the first. (I do not say it is the most importand, but it is the most characteristic of the whole tenor of this article.) Mathematics is a unity, albeit a remarkably subtle one, and we must teach mathematics to stress this. It is not true, as some claim, that all good mathematics--or even all applicable mathematics--has arisen in response to the stimulus of problems coming from outside mathematics; but it is true that all good mathematics has arisen from the then existing mathematics, frequently, of course, under the impulse of a 'real world' problem. Thus mathematics is an interrelated and highly articulated discipline, and we do violence to its true nature by separating it--for teaching or research purposes--into artificial watertight compartments. In particular, geometry plays a special role in the history of human thought. It represents man's (and woman's!) primary attempt to reduce the complexity of our three-deminsional ambience to one-dimensional language. It thus reflects our natural interest in the world around us, and its very existence testifies to our curiosity and our search for patterns and order in apparent chaos. We conclude that geometry is a natural conceptual framework for the formulation of questions and the presentation of results. It is not, however, in itself a method of answering questions and achieving results. This role is preeminently played by algebra. If geometry is a source of questions and algebra a means of answering them, it is plainly ridiculous to separate them. How many students have suffered through algebra courses, learning methods of solution of problems coming from nowhere? The result of such compartmentalized instruction is, frequently and reasonably, a sense of futility and of the pointlessness of mathematics itself.

The good sense of including applications and, where appropriate, references to the history of mathematics is surely self-evident. Both these recommendations could be included in a broader interpretation of the thrust toward an integrated curriculum. The qualification that the applications should be simple is intended to convey both that the applications should not involve sophisticated scientific ideas not available to the students--this is a frequent defect of traditional 'applied mathematics'--and that the applications should be of actual interest to the student, and not merely important. The notion of flexibility with regard to the curriculum is inherent in an integrated approach; it is obviously inherent in the concept of good teaching. Let us admit, however, that it can only be achieved if the teacher is confident in his, or her, mastery of the mathematical content. Finally, we stress as a teaching strategy the use of the hand-calculator, the minicomputer and, where appropriate, the computer, not only to avoid tedious calculations but also in very positive ways. Certainly we include the opportunity thus provided for doing actual numerical examples with real-life data, and the need to re-examine the emphasis we give to various topics in the light of computing availability. We mention here the matter of computer-aided instruction, but we believe that the advantages of this use of the computer depend very much on local circumstances, and are more likely to arise at the elementary level.

With regard to topics, we have already spoken about the link between geometry and algebra, a topic quite large enough to merit a separate article. The next two items must be in the curriculum simply because no member of a modern industrialized society can afford to be ignorant of these subjects, which constitute our principal day-to-day means of bringing quantitative reasoning to bear on the world around us. We point out, in addition, that approximation and estimation techniques are essential for checking and interpreting machine calculations.

It is my belief that much less attention should be paid to general results on the convergence of sequences and series, and much more on questions related to the rapidity of convergence and the stability of the limit. This applies even more to the tertiary level. However, at the secondary level, we should be emphasizing iterative procedures since these are so well adapted to computer programming. Perhaps the most important result--full of interesting applications--is that a sequence $\{x_n\}$, satisfying $x_{n+1} = ax_n + b$, converges to $\frac{b}{1-a}$ if $|a| < 1$ and diverges if $|a| > 1$. (For one application see [4]). It is probable that the whole notion of proof and definition by induction should be recast in 'machine' language for today's student.

The next recommendation is integrative in nature, yet it refers to a change which is long overdue. Fractions start life as parts of wholes and, at a certain stage, come to represent amounts or measurements and therefore numbers. However, they are not *themselves* numbers; the numbers they represent are rational numbers. Of course, one comes to speak of them as numbers, but this should only happen when one has earned the right to be sloppy by understanding the precise nature of fractions (see [5]). If rational numbers are explicitly introduced, then it becomes unnecessary to treat ratios as new and distinct quantities. Rates also may then be understood in the context of ratios and dimensional analysis. However, there is a further aspect of the notion of rate which it is important to include at the secondary level. I refer to average rate of change and, in particular, average speed. The principles of grammatical construction suggest that, in order to understand the composite term 'average speed' one must understand the constituent terms 'average' and 'speed' . This is quite false; the term 'average speed' is much more elementary than either of the terms 'average', 'speed', and is not, in fact, th ir composite. A discussion of the abstractions 'average' and 'speed' at the secondary level would be valuable in itself and an excellent preparation for the differential and integral calculus.

Related to the notion of average is, of course, that of arithmetic mean. I strongly urge that there be, at the secondary level, a very full discussion of the arithmetic, geometric and harmonic means and of the relations between them. The fact that the arithmetic mean of the non-negative quantities $a_1, a_2, \ldots, a_n$ is never less than their geometric mean and that equality occurs precisely when $a_1 = a_2 = \ldots = a_n$, may be used to obtain many maximum or minimum results which are traditionally treated as applications of the differential calculus of several variables-- a point made very effectively in a recent book by Ivan Niven.

Traditionally, Euclidean geometry has been held to justify its place in the secondary curriculum on the grounds that it teaches the student logical reasoning. This may have been true in some Platonic academy. What we can observe empirically today is that it survives in our curriculum in virtually total isolation from the rest of mathematics; that it is not pursued at the university; and that it instils, in all but the very few, not a flair for logical reasoning but distaste for geometry, a feeling of pointlessness, and a familiarity with failure. Again, it would take a separate article (at the very least) to do justice to the intricate question of the role of synthetic geometry in the curriculum. Here, I wish to propose that its hypothetical role can be assumed by a study of elementary number theory, where the axiomatic system is so much less complex than that of plane Euclidean geometry. Moreover, the integers are very 'real' to the student and, potentially, fascinating. Results can be obtained by disciplined thought, in a few lines, that no high-speed computer could obtain, without the benefit of human analysis, in the student's lifetime $\left((7^{10^6})^{12} \equiv 1 \bmod 13\right)$. Of course, logical reasoning should also enter into other parts of the curriculum; of course, too, synthetic proofs of geometrical propositions should continue to play a part in the teaching of geometry, but not at the expense of the principal role of geometry as a source of intuition and inspiration and as a means of interpreting and understanding algebraic expressions.

My final recommendation is also directed to the need for providing stimulus for thought. Here I understand, by a paradox, a result which conflicts with conventional thinking, not a result which is self-contradictory. A consequence of an effective mathematical education should be the inculcation of a healthy scepticism which protects the individual against the blandishments of self-serving propagandists, be they purveyors of perfumes, toothpastes, or politics. In this sense a consideration of paradoxes fully deserves to be classified as applicable mathematics! An example of a paradox would be the following: Students A and B must submit to twenty tests during the school term. Up to half term, student A had submitted to twelve tests and passed three, while student B had submitted to six tests and passed one. Thus, for the first half of the term, A's average was superior to B's. In the second half of the term, A passed all the remaining eight tests, while B passed twelve of the remaining fourteen. Thus, for the second half of the term, A's average was also superior to B's. Over the whole term, A passed eleven tests out of twenty, while B passed thirteen tests out of twenty, giving B a substantially better average than A.

4. The Elementary Curriculum

This article (like the talk itself!) is already inordinately long. Thus I will permit myself to be much briefer with my commentary than in the discussion of the secondary curriculum, believing that the rationale for my recommendations will be clear in the light of the preceding discussion and the reader's own experience. I will again organize the discussion on the basis of the 'In' and 'Out" format beginning with the 'Out' list.

b. Out (Elementary Level)

1. Teaching Strategies

Just as for the secondary level.

Emphasis on accuracy.

2. Topics

Emphasis on hand algorithms.

Emphasis on addition, subtraction, division and the order relation with fractions.

Improper work with decimals.

Commentary

The remarks about teaching strategies are, if anything, even more important at the elementary level than the secondary level. For the damage done by the adoption of objectionable teaching strategies at the elementary level is usually ineradicable, and creates the mass phonomenon of 'math avoidance' so conspicuous in present-day society. On the other hand, one might optimistically hope that the student who has received an enlightened elementary mathematical education and has an understanding and an experience of what mathematics can and should be like may be better able to survive the rigors of a traditional secondary instruction if unfortunate enough to be called upon to do so, and realize that it is not the bizarre nature of mathematics itself which is responsible for his, or her, alienation from the subject as taught.

With regard to the topics, I draw attention to the primacy of multiplication as the fundamental arithmetical operation with fractions. For the notion of fractions is embedded in our language and thus leads naturally to that of a fraction of a fraction. The arithmetical operation which we perform to calculate, say, $\frac{3}{5}$ of $\frac{1}{4}$ we <u>define</u> to be the product of the fractions concerned. Some work should be done with the addition of elementary fractions, but only with the beginning of a fairly systematic study of elementary probability theory should addition be given much prominence. Incidentally, it is worth remarking that in the latter context, we generally have to add fractions which have the same denominator--unless we have been conditioned by prior training mindlessly to reduce any fraction which comes into our hands.

Improper work with decimals is of two kinds. First, I deplore problems of the kind 13.7 + 6.83, which invite error by misalignment. Decimals represent measurements; if two measurements are to be added, they must be in the same units, and the two measurements would have been made to the same degree of accuracy. Thus the proper problem would have been 13.70 + 6.83, and no difficulty would have been encountered. Second, I deplore problems of the kind 16.1 x 3.7, where the intended answer is 59.57. In no reasonable circumstances can an answer to two places of decimals be justified; indeed all one can say is that the answer should be between 58.58 and 60.56. Such spurious accuracy is misleading and counterproductive. It is probably encouraged by the usual algorithm given for multiplying decimals (in particular, for locating the decimal point by counting digits to the right of the decimal point); it would be far better to place the decimal point by estimation.

Again, we turn to the positive side.

In (Elementary Level)

1. Teaching Strategies

As for the secondary level.

Employment of confident, capable and enthusiastic teachers.

2. Topics

Numbers for counting and measurement--the two arithmetics.

Division as a mathematical model in various contexts.

Approximation and estimation.

Averages and statistics.

Practical, informal geometry.

Geometry and mensuration; geometry and probability (Monte Carlo method).

Geometry and simple equations and inequalities.

Negative numbers in measurement; vector addition.

Fractions and elementary probability theory.

Notion of finite algorithm and recursive definition (informal).

Commentary

Some may object to our inclusion of the teacher requirement among the 'teaching strategies'--others may perhaps object to its omission at the secondary level! We find it appropriate, indeed necessary, to include this desideratum, not only to stress how absolutely essential the good teacher is to success at the elementary level, but also to indicate our disagreement with the proposition, often propounded today, that it is possible, e.g. with computer-aided instruction, to design a 'teacher-proof' curriculum. The good, capable teacher can never be replaced; unfortunately, certain certification procedures in the United States do not reflect the prime importance of mathematical competence in the armoury of the good elementary teacher.

We close with a few brief remarks on the topics listed. It is an extraordinary triumph of human thought that the same system can be used for counting and measurement--but the two arithmatics diverge in essential respects--of course, in many problems both arithmetics are involved. Measurements are <u>inherently</u> imprecise, so that the arithmetic of measurement is the arithmetic of approximation. Yes, 2 + 2 = 4 in counting arithmetic; but 2 + 2 = 4 with a probability of $\frac{3}{4}$ if we are dealing with measurement.*

The separation of division from its context is an appalling feature of traditional drill arithmetic. This topic has been discussed elsewhere [7]; here let it suffice that the solution to the division problem 1000 ÷ 12 should depend on the context of the problem and not the grade of the student.

Geometry should be a thread running through the student's entire mathematical education--we have stressed this at the secondary level. Here we show how geometry and graphing can

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*If AB = 2 ins., and BC = 2 ins., each to the nearest inch, then AC = 4 ins. to the nearest inch with a probability of 3/4.

and should be linked with key parts of elementary mathematics. We recommend plenty of experience with actual materials (e.g., folding strips of paper to make regular polygons and polyhedra), but very little in the way of geometric proof. Hence we recommend practical, informal geometry, within an integrated curriculum.

We claim it is easy and natural to introduce negative numbers, and to teach the addition and subtraction of integers--motivation abounds. The multiplication of negative numbers (like the addition of fractions) can and should be postponed.

As we have said, multiplication is the primary arithmetical operation on fractions. The other operations should be dealt with in context--and probability theory provides an excellent context for the addition of fractions. It is, however, not legitimate to drag a context in to give apparent justification for the inclusion, already decided on, of a given topic.

The idea of a finite algorithm, and that of a recursive definition, are central to computer programming. Such ideas will need to be clarified in the mathematics classroom, since nowhere else in the school will the responsibility be taken. However, it is reasonable to hope that today's students will have become familiar with the conceptual aspects of the computer in their daily lives--unless commercial interests succeed in presenting the microcomputer as primarily the source of arcade games.

But this is just one aspect of the general malaise of our contemporary society, and deserves a much more thorough treatment than we can give it here. It is time to rest my case.

Bibliography

1. Agenda for Action for the 80's, NCTM (1980).
2. New Directions in Applied Mathematics, Springer (1982), 155-163.
3. The Role of Applications in the Undergraduate Curriculum, N. R.C., Washington (1979).
4. Peter Hilton and Jean Pedersen, Approximating any regular polygon by folding paper, Mathematics Magazine 56 (1983), 141-155
5. Peter Hilton, Do we still need to teach fractions, Proc. ICME iv (1983), 37-41.
6. The Future of College Mathematics, Springer (1983).
7. Peter Hilton and Jean Pedersen, Fear No More, Addison Wesley (1983).

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LECTURE 2

THE NATURE OF PROBLEM GENERATION IN THE MATHEMATICS CURRICULUM

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The Nature of Problem Generation in the Mathematics Curriculum*

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I - Some Personal Ruminations

As Peter Hilton indicated in his address two days ago, this group is testimony to the fact that though endangered, "small" is not extinct. It also however is testimony to something more precious--to the fact that small is not incompatible with diversity of point of view, and more importantly with mutual respect for that diversity. What has been most refreshing to discover is that there is greater within group diversity (among mathematicians, mathematics educators, and school teachers) than between such groups.

I suspect that this address will reveal yet another kind of diversity and even incompatibility for which there may be slightly less tolerance: namely inconsistencies within the individual. The problem (to use a word that will invite you to view what I say recursively) is that I have thought about the subject matter of this talk for a long time. As a matter of fact, the first article in which my colleague, Marion Walter, and I ventured into the territory was published by David Wheeler when he was editing *Mathematics Teaching*. (Walter and Brown, 1969) Furthermore, not only have I recently published an article with the same theme in *FLM* (Brown, 1981), but our thinking is about to culminate in a book that draws together a decade and a half's worth of playing around with the idea of problem generation (Brown and Walter, 1983).

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*This paper is a modification of an address delivered on June 10, 1983 for the Canadian Mathematics Education Study Group at the University of British Columbia in Vancouver, Canada.

I view my problem today as one of providing some novelty such that I will not be bored by yet one more foray into a field that I have had a hand in establishing over a considerable period of time. I could of course add a small nitch to that already established tradition. After considerable reflection however, I have decided to try something more personally challenging. I will reconstruct for you a large portion of the entire terrain, but I will attempt to do so through a new set of lenses. Though I will occasionally reproduce category distinctions and examples I have previously devised, I will be approaching much of what looks like repetition from a new enough perspective so that you will have the opportunity to help unearth for me not only new potential, but the existence of inconsistencies I have alluded to earlier. For those of you who are familiar with what I have previously written and who wish to get on with the mystery, I recommend that you focus upon my comments dealing with morality and with the relationship of a problem to a situation.

II - The Rhetoric of Problem Solving

Though it was not my original intention to integrate this presentation with Peter Hilton's, it turns out that a couple of gratuitous remarks on problem solving at the end of his presentation provide a natural entree for much of what I have to say.

To begin, he points out correctly that one doesn't merely solve problems in the abstract; rather one solves specific problems. His point then is that one has to know things (and preferably a lot of things) before s/he can solve problems, and that it is a mistake to engage people in problem solving behavior before they have acquired some healthy repertoire of knowledge. The implication is that we might be better advised to familiarize students with a substantial amount of mathematics before we engage them in solving problems.

It appears to me that he has arrived at a non sequitor based upon a premise that is logically correct. That is, it certainly is true that one doesn't solve problems "in general." If I ask you what problem you are trying to solve, and you respond with, "I'm not trying to solve any specific problem at all. I'm just solving problems in general," I might have good reason to doubt not that you may be a good problem solver, but rather that you understand the meaning of problem solving in the first place.

But to say that one solves problems by working on specific "things," does not imply that those "things" themselves are "acquired" totally independently of problem solving. That is, the picture Peter Hilton has conveyed is that the following are logically or temporally related from left to right.

Being filled up with knowledge	→	problem solving

While it may be true that a "thing" (call it knowledge if you wish) is needed as a prerequisite for solving problems, it is I believe a fundamental pedagogical error to act as if those "things" can ever be acquired much as an empty vessel can be filled up. "Coming to know" anything is radically different from being filled up and the former shares some important elements with the activity of problem solving. While I will not be able to "prove" in what follows that the diagram above is essentially wrong, I hope to suggest enough through a combination of logical analysis and encouragement to join me in introspection on past learning experiences so that one of us can eventually find a careful way of accurately depicting the relationship. A pointing in that direction will furnish the background music for much of what I will be saying.

I find Peter Hilton's other comment with regard to problem solving and the curriculum more compatible with much of what I believe to be the case, though it also is in need of repair. He comments that a major difficulty with the present educational interest in problem solving is that it focuses attention at the wrong spot with regard to inquiry. That is, it leads us to focus on answers or solutions rather than upon questions. Though he may be correct in terms of existing practice, and he does capture an important truth, the situation in terms of the logic and the potential practice of problem solving in the curriculum requires considerably more "unpacking" than that brief remark would seem to warrant. Much of what follows will be an effort to explore the nature of the interrelationship and the independence of problem solving and problem generating.

In preparation for providing such linkages, I would like to dwell a little longer on problem solving per se. By the end of this section, the compelling need to relate the two will begin to emerge.

A quarter of a century ago, C. P. Snow accurately pointed out how little the two cultures--roughly the sciences and the humanities--have learned to understand each other and to gain from the wisdom they each have to offer. (Snow, 1959)

> Between the two a gulf of mutual incomprehension--sometimes . . .hostility and dislike, but most of all lack of understanding [emerges]. They have a curious distorted image of each other. . . .non-scientists tend to think of scientists as brash and boastful. . . .[They] have a rooted impression that the scientists are shallowly optimistic, unaware of man's condition. On the other hand, the scientists believe that the literary intellectuals are totally lacking in foresight, peculiarly unconcerned with their brother men, in a deep sense anti-intellectual, anxious to restrict both art and thought to the existential moment. (p. 12)

Not only are their problem solving styles different, but more importantly there are divergent views on what it means for something to be a problem in the first place, as well as what it means for something to be solved. We shall spell out explicitly some of these differences later on, but for the moment, it is worth observing that as a profession, mathematics education is almost by definition bound to the schizophrenic state of searching for and creating the "Snow-capped" bridges; for mathematics is more closely aligned with the culture and world view of science and education with that of the humanities.

As we search for a better understanding of what problem solving might be about, however we have not only neglected to build bridges, but we have tended to ignore most non-mathematical educational terrain that might be worth connecting in the first place.

In particular, we have overlooked those educational efforts in other fields which have been concerned with problem solving but have indicated that concern through a different language. Dewey's analysis of "reflective thought" and of the concept of "intelligence" would seem to offer a rich compliment to much of the problem solving rhetoric. The role of <u>doubt</u>, <u>surprise</u> and <u>habit</u> in problem solving explored by Dewey would seem to compliment much of the influential work of Polya, and would offer options we have not yet incorporated in much of our thinking about problem solving in the curriculum. (Dewey, 1920, 1933)

We have much to learn about the role of dialogue in problem solving, something we in mathematics education have tended to view in pale "discovery exercise" terms at best. Yet the use and analysis of dialogue in educational settings has been the hallmark not only of English education, but of several curriculum programs in other fields as well. "Public controversy" in the social studies in the late 60's and early 70's was a central theme around which students were taught not only to carry on intelligent dialogue, but more importantly to unearth and to discuss controversial and sometimes incompatible points of view. (Oliver and Newman, 1970). It would enrich considerably what it is we call problem solving in mathematics, if we were to entertain the possibility that for logical as well as pedagogical reasons, we might encourage not merely complementary, but incompatible perspectives on a problem or a series of problems. Furthermore such curriculum in the social studies as well as in the newly emerging field of philosophy for children might enable us to help students <u>appreciate</u> irreconcilable differences rather than to resolve or dissolve them as we are prone to do in mathematics. (Lipman, et al., 1977)

"Critical thinking is another "near relative" of problem solving that began influencing the curriculum in schools as far back as the progressive education era, and there is a considerable history of efforts to integrate different disciplines through the use of critical thought. (Taba, 1950) It is a history that is worth understanding not only because of its connection with problem solving, but because the theme is presently undergoing rejuvenation in the non-scientific disciplines much as problem solving has re-emerged in mathematics and science.

In closing this section, we turn towards one area within which the tunes of critical thinking have been re-sung recently—that of moral education. The issues that emerge here and those that we develop in the next section are part of the new (and not yet well integrated) backdrop mentioned in the first section.

First of all, we might ask why critical thinking and moral education have been joined at all. To many people, they would seem to occupy different poles. The connection hinges on our concern for the teaching of values in a pluralistic, democratic society. How do we go about such education in a public school setting without indoctrinating with regard to a particular religious or ethnic point of view? Though we might argue over whether or not it is a set of values itself and if so, why it is that such a collection is more neutral than any religious or ethnic point of view, the liberal tradition of thinking critically about whatever values one adopts does provide an entree for those concerned with morality in a pluralistic society.

Though there are a number of different kinds of programs within which moral education is taught (Lickona, 1976), most of them rely heavily upon contrived or natural dilemmas as a starting point. Our focus here will be on Kohlberg's program of moral development and education. A typical dilemma he has used for much of his research and for his deliberate program of education as well is the Heinz dilemma:

> In Europe, a woman was near death from a rare form of cancer. There was one drug that the doctors thought might save her, a form of radium that a druggist in the same town had recently discovered. The druggist was charging $2000, ten times what the drug cost him to make. The sick woman's husband, Heinz, went to everyone he knew to borrow the money, but he could only get together about half of what the drug cost. He told the druggist that his wife was dying and asked him to sell it cheaper or let him pay later. But the druggist said, "no." So Heinz got desperate and broke into the man's store to steal the drug for his wife. (Kohlberg, 1976, p. 42)

Should Heinz have stolen the drug? Based upon an analysis of longitudinal case studies to answers of dilemmas of this sort, Kohlberg has created a scheme of moral growth that he claims is developmental. Furthermore, he has created not only a research tool but an educational program around such dilemmas. It is through discussing and justifying responses to such dilemmas that students mature in their ability to find good reasons for their choices.

It is not the specific value that one chooses (e.g., steal the drug vs. allow the wife to die), but the reasons offered for the decision that places people along a scale of moral development.

At the lowest level of moral maturity, (pre-conventional) Kohlberg finds that people argue primarily from an awareness of punishment and reward. Thus someone at a lowest stage of development might claim that Heinz should not steal the drug because he would be punished by being sent to jail, or he might claim that he should steal it because his wife might pay him well for doing so. It is almost as if the punishment inheres in the action itself. At a later stage (conventional) people argue from the more abstract perspective of what is expected of you and also from the point of view of the need to maintain law and order. At the highest or stage of principled morality, one argues on the basis not of rules that could conceivably change but with regard for abstract principles of justice and respect for the dignity of human beings. Such principles single out fairness and impartiality as part of the very definition of morality.

None of these structural arguments (e.g., punishment/reward, law and order, justice) in themselves dictate what is a correct resolution of any dilemma. Rather they form part of the web that is used to justify the decisions made, and it is in listening to these reasons that Kohlberg and his followers are capable of deciding upon one's level of moral development.

Despite the fact that Kohlberg's scheme for negotiating moral development neglects to focus upon action, it is a refreshing counterpoint to a program of moral education which conceives of its role as one of inculcating specific values in the absence of reason. Nevertheless, there has been some penetrating criticism of his scheme recently--a criticism which condemns much of Kohlberg's work on grounds of sexism. That is, Kohlberg's research and ultimately his scheme for what represents a correct hierarchy of development is based upon his longitudinal research only with males. Once the scheme was created and the stages developmentally construed, Kohlberg interviewed females and concluded that their deviation from the established hierarchical scheme implied an arrested form of moral development.

Gilligan (1982) points out that the existence of a totally different category scheme for men and women not only may be a consequence of different psychological dynamics, but rather than exhibiting a logically inferior mind set, it suggests moral categories that are desperately in need of incorporation with those already derived. Compare the following two responses to the Heinz dilemma, one by Jake, an eleven-year old boy and the second by Amy, an eleven-year old girl. Jake is clear that Heinz should steal the drug at the outset, and justifies his choice as follows:

> For one thing a human life is worth more than money, and if the druggist makes only $1000, he is still going to live, but if Heinz doesn't steal the drug, his wife is going to die. (Why is life worth more than money?) Because the druggist can get a thousand dollars later from rich people with cancer, but Heinz can't get his wife again. (why not?) Because people are all different and so you couldn't get Heinz's wife again. (Gilligan, 1982, p. 26)

Amy on the other hand equivocates in responding to whether or not Heinz should steal the drug:

> Well, I don't think so. I think there might be other ways besides stealing it, like if he could borrow the money or make a loan or something, but he really shouldn't steal the drug--but his wife shouldn't die either. If he stole the drug, he might save his wife then, but if he did, he might have to go to jail, and then his wife might get sicker again, and he couldn't get more of the drug, and it might not be good. So, they should really just talk it out and find some other way to make the money. (p. 28)

Notice that Jake accepts the dilemma and begins to argue over the relationship of property to life. Amy, on the other hand, is less interested in property and focuses more on the interpersonal dynamics among the characters. More importantly, Amy refuses to accept the dilemma as it is stated, but is searching for some less polarized and less of a zero sum game.

Kohlberg's interpretation of such a response would imply that Amy does not have a mature understanding of the nature of the moral issue involved--that she neglects to appreciate that this hypothetical case is attempting to test the sense in which the subject appreciates that in a moral scheme life takes precedence over property. Gilligan on the other hand in analyzing a large number of such responses has concluded not that the females are arrested in their ability to move through his developmental scheme, but that they tend to abide by a system which is orthogonal to that developed by Kohlberg--a system within which the concepts of caring and responsibility rather than justice and rights ripen over time.

Gilligan (1982) comments with regard to Amy's response:

> Her world is a world of relationships and psychological truths where an awareness of the connection between people gives rise to a recognition of responsibility for one another, a perception of the need for response. Seen in this light, her understanding of morality as arising from the recognition of relationship, her belief in communication as the mode of conflict resolution, and her conviction that the solution of the dilemma will follow from its compelling representation seem far from naive or cognitively immature. (p. 30)

The difference between a "Kohlbergian" and a "Gilligannish" conception of morality is well captured by two different adult responses to the question, "what does morality mean to you?". (Lyons, 1983) A man interviewed comments:

> Morality is basically having a reason for doing what's right, what one ought to do; and, when you are put in a situation where you have to choose from amongst alternatives, being able to recognize when there is an issue of "ought" at stake and when there is not; and then. . .having some reason for choosing among alternatives.(p. 125)

A woman interviewed on the same question comments:

> Morality is a type of consciousness, I guess a sensitivity to humanity, that you can affect someone else's life. You can affect your own life and you have the responsibility not to endanger other people's lives or to hurt other people. So morality is complex. Morality is realizing that there is a play between self and others and that you are going to have to take responsibility for both of them. It's sort of a consciousness of your influence over what's going on.(p. 125)

While Gilligan and her associates do not claim that development is sex bound in such a way that the two systems are tightly partitioned according to gender, they do claim to have located a scheme that tends to be associated more readily with a female than a male voice. Behind the female voice of responsibility and caring, some of the following characteristics appear to me to surface:

1. A context boundedness,
2. A disinclination to set general principles to be used in future cases,
3. A concern with connectedness among people.

Though not all of these characteristics are exhibited in Amy's response, they do appear in interviews with mature women. <u>Context boundedness</u> represents a plea for more information that takes the form not only of requesting more details (e.g., what is the relationship between husband and wife?) but of searching for a way of locating the episode within a broader context. Thus unlike men, mature women might tend to respond not by trying to resolve the dilemma, but by exhibiting a sense of <u>indignation</u> that such a situation as the Heinz dilemma might arise in the first place. Such a response might take the following form: The question you should be asking me is 'What are the horrendous circumstances that caused our society to evolve in such a way that dilemmas of this sort could even arise--that people have learned to miscommunicate so poorly'?"

The second characteristic I have isolated above, is an effort to attempt to understand each situation in a fresh light, rather than in a legalistic way--i.e., in terms of already established precedent. Connected with context boundedness it is the desire to see the fullness of "this" situation in order to see how it might be <u>different from</u> (and thus require new insight) rather than compatible with one that has already been settled.

With regard to the third characteristic, conflict is less a logical puzzle to be resolved but rather an indication of an unfortunate fracture in human relationships--something to be "mended" rather than an invitation for some judgement.

In the next section we turn towards a consideration, in a rather global way, of how it is that a Gilliganish perspective of morality might impinge on the study of mathematics. While we have not yet drawn any explicit links, it is not difficult to intuit not only that it threatens the status quo but that it sets a possible foundation for the relationship of problem generation to problem solving. Though we shall focus upon the findings from the field of moral education, we do not wish to lose sight of some of the other humanistic of areas of curriculum from which mathematics education might derive enlightenment.

Some of what we have alluded to earlier in this section will form the background music for what follows:

III - Kohlberg vs. Gilligan: The Transition from Solving to Posing

It surely appears that problem solving in mathematics education has been dominated by a Kohlbergian rather than a Gilliganish one. Gilligan herself has an intuition for such a proposition, when she comments with regard to Jake's response to the Heinz dilemma:

> Fascinated by the power of logic, this eleven-year old boy locates truth in math, which he says is "the only thing that is totally logical." Considering the moral dilemma to be "sort of like a math problem with humans," he sets it up as an equation and proceeds to work out the solution. Since his solution is rationally derived, he assumes that anyone following reason would arrive at the same conclusion and thus that a judge would consider stealing to be the right thing for Heinz to do. (p. 26,7)

The set of problems to be solved as well as the axioms and definitions to be woven into proofs are part of "the given"--the taken-for-granted reality upon which students are to operate. It is not only that the curriculum is "de-peopled" in that contexts and concepts are for the most part presented ahistorically and unproblematically, but as it is presently constituted the curriculum offers no encouragement for students to in a respectable way move beyond merely accepting the non-purposeful tasks.

Furthermore, rather than being encouraged to try to capture what may be unique and unrelated to previous established precedent in a given mathematical activity (the legalistic mode of thought we referred to as the second characteristic behind Gilligan's analysis of morality as responsibility and caring), much of the curriculum is presented as an "unfolding" so that one is "supposed" to see similarity rather than difference with past experience. It is commonplace surely in word problems to tell people to ignore rather than to embellish matters of detail on the ground that one is after the underlying structure and not the "noise" that inheres in the problem.

In so focusing on essential isomorphic features of structures, the curriculum tends not only to threaten a Gilliganish perspective, but as importantly, it supports only one half of what I perceive much of mathematics to be about. That is mathematics not only is a search for what is essentially common among ostensibly different structures, but is as much an effort to reveal essential differences among structures that appear to be similar. (See Brown, 1982a)

With regard to context boundedness, there is essentially no curriculum that would encourage students to explicitly ask questions like:

- What purpose is served by my solving this problem or this set of problems?
- Why am I being asked to engage in this activity as this time?
- What am I finding out about myself and others as a result of participating in this task?
- How is the relationship of mathematics to society and culture illuminated by my studying how I or other people in the history of the discipline have viewed this phenomenon?

Elsewhere (Brown 1973, 1982) I have discussed how I first began to incorporate such reflection as part of my own mathematics teaching, and I shall have other illustrations of so doing in this paper. There are a number of serious questions that must be thought through, however, before one feels comfortable in encouraging the generation and reflection of the kinds of questions indicated above. We need to be asking ourselves whether or not that kind of reflection represents respectable mathematical thinking. In addition we ought to be concerned about the ability of students to handle that thinking in their early stages of mathematical development.

It is interesting to observe that though the wedge is being provided to integrate mathematics with other fields, the "real world" applications seem to be narrowly defined in terms of the scientific rather than the humanistic disciplines. In particular questions of value or ethics are essentially non-existent. That is particularly surprising in light of the fact that a major rationale for relating mathematics to other fields seems to be that such activity may enable students to better solve "real world" problems that they encounter on their own. I know of essentially no "real world" problems that one decides to engage in for which there is not embedded some value implications.

McGinty and Meyerson (1980) suggest some steps one might want to take to develop curriculum for which value judgements are an explicit component. They begin with a problem like the following:

> Suppose a bag of grass seed covers 400 square feet. How many bags would be needed to uniformly cover 1850 square feet? (p. 501)

So far so dull. It is not only that for many students the above would not constitute a problem (in the sense of answering the question), but more importantly it lacks any reasonable conception of context boundedness. The authors, however, go on to suggest inquiry that is more "real worldish" than most of the word problems students encounter. They ask:

> Should the person buy 5 bags and save the leftover--figuring prices will rise next year? Buy 5 bags and spread it thicker? Buy 4 bags and spread it thinner? (p. 502)

Once we become aware of ethical/value questions as a central component of decision making, it is clear that there is much more we might do in the way of generating problems for students as well as encouraging them to do so on their own. One of the <u>au courant</u> curriculum areas is probability and statistics. As a profession, we correctly appreciate that we need to do more to prepare students to operate in an uncertain world, wherein one's fate is not sown with the kind of exactitude that much of the earlier curriculum has implied. In creating such a curriculum, however, we continue to give the false illusion that mathematical competence is all that is required to decide wisely. Compare <u>any</u> probability problem (selected at random of course) from any curriculum in mathematics with the following probability problem:

> A close relative of yours has been hit by an automobile. He has been unconscious for one month. The doctors have told you that unless he is operated upon, he will live but remain a vegetable for the rest of his life. They can perform an operation which, if successful, would restore his consciousness. They have determined, however, that the probability of being successful is .05, and if they fail in their effort to restore consciousness, he will certainly die.

What counsel would you give the doctors? One could clearly embed the above problem in a more challenging mathematical setting, for example, setting up the conditions that would have enabled one to arrive at the .05 probability (or perhaps modifying it so that outer limits are set on the probability of survival) but nevertheless, it is such ethical questions in many different forms that plague most thinking people as they go through life making decisions.

Is such problem generation on the part of the teacher or student an ingredient of mathematical thought? I do not think the answer is clear. There is nothing god-given and written in stone that establishes what is and is not part of the domain of mathematics, and clearly what has constituted legitimate thinking in the discipline has changed considerably over time. I am not familiar enough with the sociology of knowledge to know what kinds of forces other than logical ones have been responsible for driving people to reconceptualize the discipline of mathematics, but even if questions of the kind we have been raising in this section would move us in directions that are at odds with the dominant and respectable mode of mathematical thought, it is worth appreciating that as educators we have a responsibility to future citizens that transcends our passing along <u>only</u> mathematical thought. The latter appears to me to be a very narrow view of what it means to educate. In realizing that only a very small percentage of our students will be mathematicians, we have not adequately explored our obligation to those who will not expand the field <u>per se</u>. We have mistakenly identified our task for the majority as one of "softening" an otherwise rigorous curriculum. What may be called for is an ever more intellectually demanding curriculum, but one in which mathematics is embedded in a web of concerns that are more "real world" oriented than any of us have begun to imagine.

It is worth observing that such complications of mathematical thinking may in fact pose a major threat to a concept that we have begun in recent years to revere--that of mathematization. In attempting to find reason to believe that children can indeed function as mathematicians (as opposed to exhibiting routine imitative skills), David Wheeler (1982) looks towards exceptional cases of mathematical precocity. He comments:

> I don't see children however exceptional function as historians, or as lawyers, or as psychologists, for instance, since these are extremely complex functionings that involve subtle relationships between (sic) several frames of reference. But I would hypothesize that mathematics belongs with art, music, writing and possibly science, as one of a class of activities that require only a particular kind of response to be made by an individual to his immediate, direct experience. (p. 45)

While I would certainly not wish to pit mathematization, as Wheeler describes it, against the mindless symbol pushing that represents its polar opposite, I believe that as educators we are obligated to push the bounds of complicating that discipline in an effort to engage the minds of students in directions that define their humanity.

IV - Down From A Crescendo

How do we descend from the heights and perhaps the overinflated language which concluded the previous section? Perhaps one way is to take stock of where we have been led and to try to sharpen the implications that might follow. The confrontation between Kohlberg and Gilligan has served two purposes that appear on the surface to be very different. First of all, we have used the challenge of Gilligan's research to point out that there is a world view that has achieved empirical expression with regard to issues of morality but which is worth taking seriously in other domains as well. Moving beneath the concepts of caring and responsibility established by Gilligan, we find dimensions that are not strictly moral in character but which deal with purpose, situation specificity (a non-legalistic mode) and people connectedness. We have suggested that very little of the existing mathematics curriculum caters to those characteristics, and in fact the dominant mode caters to their opposite.

Secondly, we have not only used Gilligan in contrast to Kohlberg to establish broad categories within which the present curriculum is deficient, but we have pointed out that what the two perspectives have in common--namely a concern with morality--represents a field of inquiry that may be as important to integrate with mathematical thinking as are the more standard disciplines that form the backbone of more conventional applications.

Both of these perspectives have potentially revolutionary implications. They not only suggest the need for both teacher and student to incorporate a more serious problem generating perspective (including the broad types of questions raised at the beginning of the previous section) as an essential ingredient of problem solving, but they have the potential to infect every aspect of mathematics education from drill and practice, to an understanding of underlying mathematical structures.

Our goal for the remainder of this paper will be the more modest one of making a case for the inclusion of problem generating strategies within the curriculum. I will for the most part be drawing upon and integrating ideas that I have previously developed. While I will make minimal explicit reference to the Gilligan perspective, I believe it is possible to view much of what follows as being derived from what I have referred to as the underlying components of caring and being responsible. The joining of links explicitly in other mathematics education arenas is a task to be left for another time (and perhaps another person).

V - Our Knee-Jerk Solving Mentality

Any field of inquiry establishes a common language among its investigators. The same kind of phenomenon is exhibited among friends, lovers and members of a family. It is frequently possible to determine the extent to which you are in fact an "outsider" by the degree to which you are incapable of understanding the short-circuiting of language among participants. There is certainly good reason for members of an "in-group" to engage in such short-circuiting behavior. In addition to merely increasing efficiency of communication, there are important psychological and sociological bonds established through such behavior.

Nevertheless, we sometimes pay a price for the common language we establish. That is, in focusing on common understanding, we not only leave out other perspectives, but we may be <u>unaware</u> of what we are leaving out. The specialization that results from such behavior not only may leave us unaware of what we have left out, but worse than that, we may even lose our ability to incorporate those awarenesses within our world view even when they are pointed out to us.

As educators, it is worth taking stock every so often to examine explicitly what we are leaving out in the common language we are establishing with our students. Such an instance occurred a number of years ago at which point Marion Walter and I were team teaching a course on problem solving. We were doing work in number theory, and were hoping to derive a formula to generate primitive Pythagorean triplets. We began the lesson by asking:

$x^2 + y^2 = z^2$. What are some answers?

Responses began to flow, and students responded with:

3, 4, 5
5, 12, 13
8, 15, 17

After a while, a smile broke out on the face of a student who gave us:

$1, 1, \sqrt{2}$.

A few more "courageous" and humorous responses then were suggested like:

$$-1, -1, \sqrt{2}.$$

Marion and I then jokingly reprimanded the "deviants," and proceeded to explore what we were about in the first place--a search for a generating formula. After class, however, we began to talk to each other about the incident. It hit us very hard that the "deviants" were beginning to appreciate something that has occupied a considerable part of our collective energy for the past fifteen years.

What struck us was that:

$x^2 + y^2 = z^2$. What are some answers?

has a kind of foolishness about it that derives from the closed position of a common language with its unspoken but built in assumptions. Notice that $x^2 + y^2 = z^2$ is not even a question. How can one come up with answers?

Yet the students dutifully did come up with answers, because they carried along a host of assumptions that we in fact have trained (implicitly) them to accept. They assumed (at least at the beginning) that the symbolism had connoted that the domain was natural numbers. Furthermore they assumed that the symbolism was calling for something algebraic; and within that context they assumed that we were searching for instances that would make an open sentence true.

As soon as we began to appreciate that "the deviants" had begun to appreciate something we had not seen, we realized that there was a whole new ball game at stake. We had not realized at the time that in expanding this concept for this class, we were opening Pandora's box.

In realizing that we had implicitly assumed that the domain was natural numbers, we encouraged students to ask such new questions as:

For what <u>rational</u> numbers x, y, z is it true that $x^2 + y^2 = z^2$?

Realizing that we had implicitly assumed that we were searching for true instances of the open sentence, we encouraged students to ask such new questions as:

For what natural numbers is it true that $x^2 + y^2 = z^2$ is "almost" true? (e.g., 4, 7, 8 misses the equality by 1).

Realizing that we had implicitly assumed that the question was algebraic, the students began to ask a host of geometric questions that derived from connotations of the algebraic form.

What followed immediately was one of the most intellectually stimulating units that either of us had previously experienced with our students, and what dawned eventually on all of us was something that has had a lasting effect.

First of all, we began to appreciate that such deviations from standard curriculum are not mere frills. That is, in exploring such questions as the "almost" primitive Pythagorean triplet question, all of us gained a much clearer understanding of what the actual primitive Pythagorean triplet question was in fact about--not only from the point of view of statement but of proof as well.

Secondly, and more importantly, we began to realize that an implicit part of the common language we share with students is one which focuses upon and points so strongly towards the search for solutions and answers, that we continue to search for answers even when no question is asked at all! We were thus launched on our journey to try to understand the role of problem generation is the doing of mathematics.

VI - Posing and Deposing: A First Step

I am beginning to appreciate an important aspect of what is behind an understanding of the role of posing problems that I have not seen before, despite the fact that I have referred in much of my writing to examples within which this issue is embedded. For a number of years, educators have appreciated that there might be considerable value in giving students not problems to solve but situations to investigate. Higginson (1973), for example, locates a number of characteristics of what he refers to as "potentially rich situations." Situations are much "looser" than problems, and situations themselves do not ask built-in questions. It is the job of the student to create a question or pose a problem. Geoboards, Cuisenaire Rods, polyominoes are all examples of situations, but situations need not be concrete materials; they can be abstractions as well.

What I have recently (in preparing for this talk) begun to appreciate is that the pedagogical issue is much deeper and more interesting than that of merely creating rich situations to investigate. The issue is even more complicated than providing both mechanisms and an atmosphere within which problems might be isolated from situations. Rather the pedagogical task is one of enabling all of us to appreciate the differences between a problem and a situation, and of finding ways to move from one to the other.

The task of so moving is neither mechanical nor easy. That it sometimes takes a very long time to appreciate that a situation implies a problem is something that most parents experience through much of their child rearing. That problems can be neutralized (or de-posed as the title of this section playfully suggests) is something that may be equally difficult to appreciate. Those of us who realize that we have been asking the wrong questions realize implicitly the need to move from a problem to a situation before re-posing the problem.

Consider the example of a "female response" to the Heinz dilemma which asserts with indignation that the problem is not one of stealing or not stealing the drug, but rather one of figuring out how we even evolved as a society such that such choices would have to be made (and one of figuring out how to reconstruct society). Here is a clear case of first neutralizing a problem before re-posing it. It was necessary to delete the question (Should Heinz steal the drug?) before moving towards a re-posing of the problem.

It is not only that there is value in having students actually move in both directions--from situations to posing and from posing to de-posing--but it is also worth designing curriculum which exhibits the difficulties people had in making such moves on their own in the history of the discipline. We have the potential to learn a great deal about the relationship of a discipline to the culture from which it emerges as we study those problems that could not be perceived as situations.

An obvious example in the history of mathematics is that of efforts over several centuries to try to prove the parallel postulate. Consider the following formulation of the question:

> How can you prove the parallel postulate from the other postulates of Euclidean geometry?

We know now that a great deal of the history of mathematics was written as nineteenth century mathematicians began to appreciate that the difficulty in solving the problem was that a wrong question was being posed. In some implicit sense, Lobachevsky and his colleagues at the time had in fact to "neutralize" the problem enough first to get clearly at the situation from which it derived (the postulates of Euclidean geometry) and then to reformulate the question so as to delete the deceptively innocent word "How" in the posing of the problem.

The need to re-pose a problem by first neutralizing it is not only revealed through frustrated efforts at solving problems, but is an aesthetic issue as well, and an issue that is worth incorporating explicitly in curriculum within which the Gilliganish concept of context boundedness is taken seriously. Consider the case of efforts to prove the four color conjecture--roughly that for any conventional map, four is a sufficient number of colors to establish and to appropriately demarcate boundaries. Until recently the problem was "merely" to prove or disprove that conjecture. Only after a computer proof was produced which featured a very large number of special cases did mathematicians begin to realize that they had not adequately posed the problem. Feeling that a computer proof was blind to underlying structure and in fact illuminated very little of "the mathematical essence" of the problem, many mathematicians realized the need to state the problem in such a way that "ugly" proofs would not count as solutions.

Such re-posing of the four color problem reveals something not only about the present attitude of many mathematicians with regard to the computer, but just as importantly, it unearths some fundamental epistemological issues--issues that more clearly locate knowledge within an aesthetic realm.

From a pedagogical point of view, it is particularly enlightening to engage students in a discussion of the relationship of a situation to a problem. I have a modest example. Several years ago my son, Jordan, came to me to tell me that he did not understand the "ambiguous case" in trigonometry, i.e., those circumstances under which a triangle is determined by an angle, another angle and a side not included between the angles.

I began my discussion with him by asking him to recall how in geometry, he had investigated those conditions under which a triangle was determined. Jordan looked very puzzled and told me that I was mistaken; they had never investigated the determination of a triangle. Instead they had proven things about two triangles being congruent if A.S.A. = A.S.A. and so forth.

What was taking place here is very interesting from the point of view of relating a problem to a situation. Jordan had in fact viewed an entire unit of work more as a <u>situation</u>, while I had viewed it as a <u>problem</u>. That is, though he had an arsenal of congruence theorems at his disposal to respond to any request to prove two triangles congruent, he did not see this ammunition as providing answers to what I saw to be the fundamental problem of discovering those conditions under which a triangle is determined. As I reviewed his text, I understood why he saw a situation in what I saw to be a problem. The book had in fact never distinguished between an underlying problem (determining a triangle) and a collection of exercises to give one experience in handling a problem that had been solved by the famous congruence theorem. In fact, the practice exercises had become the fundamental concept--a phenomenon I am beginning to believe is more widespread than I had thought, and a consequence most likely of the essentially plagiaristic spirit that governs text book writing.

The interesting irony in this case is that the difference between my perception and Jordan's regarding what those congruence theorems were all about, was not revealed in Jordan's performance in geometry at all. One can frequently accurately answer questions and even solve difficult problems without seeing the context within which those problems are embedded.

Thus, it would seem to be a very wise pedagogical ploy to move not only from situation to problem and back for topics that are relatively small (e.g., de-pose a theorem such as "The base angles of an isosceles triangle are congruent"), but to do so for entire units as well. Teachers as well as students would find it enlightening to discover the areas of agreement and divergence of opinion regarding the problem/situation status of a unit or perhaps even of a course.

In closing this section, I would like to comment on an interesting potential difficulty relating situation to problem that Peter Hilton alluded to in his talk. He mentioned that proper selection of problems is critical in designing curriculum for one does not want to give problems to students that they are not prepared to handle. His comment appears on the surface to be a threat to the activity of posing and de-posing problems. That is, what happens if in the creation of a problem from a situation, a student defines a problem that we know is beyond his/her ability to handle?

There are some interesting assumptions embedded in the above question. First of all, it is not necessarily the case that students need to try to solve problems they pose. The activity of posing itself in the absence of efforts to solve may be illuminating both to students and teachers. In a sense we find out as much of value about ourselves by attending to the kinds of questions we ask as we do by the solutions we attempt.

Secondly, if we think of an entire class as a unit, for the kinds of activities suggested in this section, it is not necessarily the case that the same person who poses a problem need be obligated to try to solve it. In fact we may discover the potential for unexpected collaboration among those who pose and those who attempt solutions. We do not know very much at all about the relationship of the talent of posing and solving, but it perhaps is worth taking a clue from the work of Getzels and Jackson (1961) in which they find reason to conclude that beyond a certain point, intelligence and creativity may not be as closely related as one might suppose.

But there is another consideration that cuts deeper than those we have mentioned so far. That is, what does Hilton imply students are expected to do who are prematurely challenged? It appears that they may be incapable of <u>solving</u> problems that either we (as teachers) or they pose. Such an expectation may be a short-sighted one from an educational point of view however. Along with our newly discovered appreciation for the role of approximation and estimation, ought to come an appreciation for partial solutions as a respectable activity. We need not necessarily expect a complete solution for every problem investigated. In addition, I am not clear on what it is that is lost if students attempt to solve a problem and cannot even come up with partial solutions. Suppose they cannot even identify or isolate lemmas that might help them along the way. I can imagine a great deal of valuable personal and intellectual insight that might emerge through a discussion of what may account for inability of students at a particular point in time to make headway in solving particular problems. A teacher who keeps an ear to the ground might possibly even learn something of the students conception of the subject matter, proof, mathematics and the relationship of mathematics to culture by listening carefully to what counts as a reason for failure to make headway.

VII - The Act of Posing: Logic and Pedagogy

In relating problem posing to the creation of situations, we have, beneath the surface bumped up against the relationship of problem posing to problem solving. After all, it was due to an inability to solve the parallel postulate problem that a situation was revealed which was in need of reformulation. Problem generation and problem solving are intimately connected, however, even when things do not go awry. Below we discuss their intimate logical connection. In the two subsections that follow the one below we shall look more closely at pedagogical strategies for engaging in problem posing--one mild and the other radical. Much of what I will be analyzing in this section has appeared in disparate sources, and I view the task here as one primarily of consolidating that material. For that reason this section will be briefer than the others (thank God!) and the reader's attention will be drawn to relevant references for expansion of the points alluded to.

Logical Connections With Solving: Being Gracious and Accepting

Consider the following two problems:

(1) A fly and train are 15 km. apart. The train travels towards the fly at a rate of 3 km/hr. The fly travels towards the train at a rate of 7 km/hr. After hitting the train, it heads back to its starting point. After hitting the starting point, it once more heads back toward the train until they meet. The process continues. What is the total distance this fly travels?

(2) Given two equilaterial triangles, find the side of a third one whose area is equal to that of the sum of the other two.

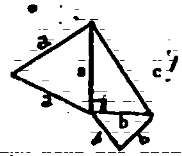
The first problem reveals in a dramatic way something that is true but less obvious in the solution of any problem. If you have not seen this problem before, let it sit for awhile, or perhaps share it with a fifth grader. If the wind is blowing properly, you will come upon an insight that will most likely jar and inspire you. Without giving the ball game away completely, let me suggest that an insightful and non-technical solution depends upon your asking a question that has not been asked in the problem at all. Though there are many different ways of asking the question as well as many questions to ask, something like the following will most likely be revealing:

> What do I notice if I focus not on the fly as requested, but on the train instead?

What is needed in the solution of this problem is some effort at posing a new problem within the context of accepting and trying to solve a given problem. Whether or not such problem posing is <u>always</u> needed in the solution of a problem is an interesting and debatable question. I believe that such problem generation is always needed, but I also believe that the analysis of the assertion very much hinges on how it is that one defines a problem in the first place. (See Brown, 1981a; Brown and Walter, 1983 for additional discussion of this point)

The second problem reveals another interesting intimate connection between problem solving and problem generating. The solution depends (an illustration of what we have said above) upon how it is that the problem itself is re-defined. If, however, you assume that sides and their lengths can be distinguished from each other (something that is not necessary in the solution of the problem), then if the lengths of the sides of the first two triangles are a and b respectively, we can prove without too much fanfare that the length of the third side c is equal to $\sqrt{a^2 + b^2}$.

Now in one sense we have solved the original problem. In another sense, however, we have only begun to solve it. Most people who come upon the solution, $c = \sqrt{a^2 + b^2}$ are taken aback. The point is that it smells as if this is an interesting and unexpected connection (as a matter of fact one which now enables one to solve the problem without associating the sides with their lengths). The fact that the relationship is a Pythagorean one, indicates that we can find the third side as suggested below:



Most mathematicians who have not seen this problem before find themselves headed in an almost compulsive search for what is happening. They are driven by some variation of the question:

> I know areas are additive for the squares on the sides of a right triangle, but *why* are they additive for equilateral triangles as well.

What this example illustrates very nicely is that a proof or a solution in itself does not always reveal *why* things operate as they do. Something more is needed, and in this case that something more begins with a question.

Though it is surely the case that the alleged solution of any problem always has further implications that one may assert as a problem or a question, one is not necessarily driven to do so in all problems with the same kind of fervor as in this case. (See Walter and Brown, 1977 and Brown and Walter, 1983 for an elaboration of this discussion)

There are pedagogical implications that flow from these relationships between posing and solving problems. Students are not always aware of the questions they may have implicitly asked themselves in coming up with the solution to a problem, and there might be value in encouraging them to explicitly see what they have done. At the other end of the spectrum, students may not at all be aware of additional questions they "need to" or might ask after they have supposedly solved a problem.

On Strategies For Posing: An Accepting Mode

It is one thing to suggest that problem posing is worthwhile, or even necessary; it is another to be able to do so. We shall in this subsection suggest several strategies for posing problems, some of which are well discussed in the literature, and some of which represent new directions. In this subsection and the next, we shall look at the activity of problem generation in a mode that is somewhat isolated from that of solving a problem that has already been stated. In so doing, we return to situations as a starting point. Much of what we do here might be appropriate to apply to the activity of solving an already stated problem as well. (See Brown and Walter, 1983 for an elaboration of these two subsections)

What are the "things" that situations are made of? Among possible candidates are the following:

1. concrete objects like Cuisenaire Rods and the Tower of Hanoi
2. abstract "things" like
   (a) Isosceles Triangles
   or
   (b) Nine Supreme Court Justices each shaking hands with each other
3. data like
   (a) Primitive Pythagorean Triplets generated by the relationships $x^2 + y^2 = z^2$ (like: 3,4,5; 5,12,13; 7,24,25)
   or
   (b) 5,12,19,26,33, . . .
4. theorems or postulates like the Fundamental Theorem of Arithmetic (every number can be expressed uniquely as a product of primes)

There are surely more kinds of "things" that one might use as a starting situation, but the above should serve the purpose of enabling to see how the directions we might look towards in generating questions.

(i) Estimation/Approximation

Here is a category with which we are all familiar, though we tend not to make as much use of it in practice as we might. Given phenomenon 2(b) for example, most people with a little knowledge will ask: How many handshakes are there? Of course it is just as illuminating (for some purposes) a question to ask: About how many handshakes are there?

(ii) Internal and External Views of a Thing:

Given situation 2(a), most people will ask the rather familiar question: What can you say about the base angles? Some people might extend the base and ask about the external angles. Compare these kinds of questions with one like:

> How many isosceles triangles can you join to form the hub for a bicycle wheel?

How does the above question differ from the other isosceles triangle questions? It is worth pointing out that while the first set focuses on the internal workings of the phenomenon, the one dealing with the hub takes the isosceles triangle in its entitety and relates it to something else. Much of our standard curriculum is focused on an internal view of objects and relatively little takes as its starting point the object as a whole.

(iii) The Particular and the Specific

Here is a theme that is particularly salient in terms of a Gilligan perspective. Take a look at 3 above and pose some problems.

Our enchantment with abstraction and generalizability frequently blinds us from seeing the uniqueness of what is before us. Most people shown 3(a) and (b) will pose a problem that attempts to reveal some covering law that will generate all the terms. A careful look at data, however, frequently suggests that there is more to see that might be equally as appealing. Consider the following for example with regard to 3(a):

> Each triplet has at least one member divisible by 3, by 4, and by 5. Will that hold in general?

The above is clearly not a question that would arise if our focus were upon the more abstract Pythagorean relationship.

Take another look at 3(b). What questions arise from a careful look at the data beyond a search for some general algebraic generating formula?

(iv) On Pseudo-History

Many teachers wish they knew more about the history of mathematics so that they might be better able to motivate the subject. What is not well appreciated, however, is that a great deal of intellectually stimulating thought can flow from an effort on the part of students as well as teachers to engage in what I call pseudo-history (Brown, 1978 as well as Brown and Walter, 1983). As an example, consider the following kind of question conceivably generated by 4.

> What might have been responsible for getting people to look at products of primes?

We can, for example, imagine a mathematics community that focused originally on expressing any given number as the sum of other numbers. What might have moved them to look at products instead?

These are surely not the only categories for generating problems while at the same time maintaining an accepting view towards the beginning situation. They do, however, represent a start, and with the exception of (i) tend not to be given much curriculum consideration. It would be a valuable contribution to expand both the list of "things" upon which one might generate questions as well as the categories one might look towards in the generation of questions.

Posing As An Adolescent

In this subsection we further expand the concept of problem generation by selecting a situation in a mode that is more reminiscent of adolescent rebellion than is the previous subsection. Though perhaps the most intriguing, this is the aspect of problem posing that Marion Walter and I have written more about than any other, and I therefore will restrain my natural tendency towards verbosity—with the suggestion that you refer to relevant pieces cited if you wish further elaboration.

The concept of challenge, threat or adolescent rebellion is well captured by Hofstadter (1982) when he comments:

> George Bernard Shaw once wrote (in Back to Methuselah): "You see things; and you say 'Why?' But I dream things that never were; and I say 'Why not?'" When I first heard this euphuism, it made a lasting impression on me. To "dream things that never were"--this is not just a poetic phrase but a truth about human nature. Even the dullest of us is endowed with this strange ability to construct counterfactual worlds and to dream. Why do we have it? What sense does it make? How can one dream or even "see" what is visibly not there?. . .Making variations on a theme is really the crux of creativity. On the face of it the thesis is crazy. How can it possibly be true? Aren't variations simply derivative notions, never truly original creations? (p. 20) . . .Careful analysis leads one to see that what we choose to call a new theme is itself always some kind of variation, on a deep level of earlier themes (p. 29).

One can start with a definition, a theorem, a concrete material, data, or any other phenomenon and instead of accepting it as the given to be explored, one can challenge it and in the act create a new "it."

> Consider for example the definition of a prime number:
> A natural number is prime if it has exactly two different divisors.*

Now the "natural" inclination of the standard curriculum is to use that bit of information to prove or show all kinds of things. An adolescent rebellion on the other hand might generate a host of questions like:

- What's so special about numbers that have exactly two different divisors? What kinds of numbers have exactly three divisors?
- Why do we focus on divisor? Can we find numbers that have exactly two different elements to form a sum?
- Why are we focusing on different divisors? Can numbers have the same divisor twice?
- Why do we focus on natural numbers? Suppose we look instead at fractions or the set of odd integers.

I shall not continue with the list of such questions that can be generated to challenge rather than accept the concept of prime number. (See Brown, 1978, 1981 for a thorough development especially of the last question) Let me merely indicate that such activity has a built-in kind of irony, for it is in the act of "rebellion" that one comes to better understand the "thing" against which one rebels. In that sense challenging "the given" as a strategy for problem generating has the potential to be viewed as a less radical departure from standard curriculum than one might otherwise believe.

Marion Walter and I have taken the insight of challenging the given and created a scheme which we call "what-if-not." A number of people both within the CMESG group and outside of it, have derived some fascinating and imaginative concepts by employing the scheme. Though it is possible to approach that scheme in an overly mechanistic manner, it is also something that can be done with taste.

Suppose one wishes to do a "what-if-not" on the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, . . .

For the first stage of the scheme, one lists the attributes of "the thing," without worrying about such matters as completeness, repetition, elegance of statement, independence of statements and so forth. Thus we might list among the attributes:

1. The sequence begins with the same first number.
2. The first two numbers are 1.
3. If we do something to any two successive terms, we get the next number in the sequence.
4. The something we do is add.

At the second stage, we do a "what-if-not" (hence the name of the scheme) on one of the attributes. For example, suppose we do a "what-if-not" on number 2 above; if it is not the case that the first two numbers are 1, we ask what they might be? Obviously we could select many alternatives to 1 and 1 as the starting numbers. Suppose we chose 3 and 7.

At the next stage, we ask some new set of questions about the modified phenomenon. Suppose we begin by asking what the new sequence would look like. To continue the process, we finally engage in the kind of activity which most people incorrectly assume is the essence of mathematics--namely we analyze or try to answer the question. Thus, if we maintain the essential definition of the original sequence (something we need not necessarily feel obligated to do), we would get:

3, 7, 10, 17, 27, 44, 71, . . .

Moving back to the stage of asking some new set of questions, we might ask:

- Is there an explicit formula to generate the nth term of the sequence?
- How do properties of this sequence compare with those of the original one?

An analysis of these questions reveals some very fascinating jewels. People who are familiar with properties of the original Fibonacci sequence, in analyzing the second question above, most likely would look (among other things) at ratios of succeeding terms. Choosing smaller to larger adjacent terms, we would get:

.42, .70, .588, .708, .614, .62,

Something smells (as in the equilateral triangle example in the previous subsection) peculiar. We are arriving at ratios that appear to be very close to the "golden ratio" (approximately .618)--something we expect from the original Fibonacci sequence. Why is that happening?

In analyzing the question above, one is thrown back towards an analysis of the original phenomenon--as we indicated above.

We have barely begun to see the wealth of surprising results in making use of the "what-if-not" strategy on the Fibonacci sequence. (See Brown, 1976 and Brown and Walter, 1983 for a more detailed discussion) In this brief sketch, however, we implied the value both of carefully employing the various stages of the "what-if-not" strategy and of interrelating them as well.

In closing, it is worth pointing out that despite efforts to mechanize the stages, the process tends to elude a computerized mentality, for it is frequently the case that in the absence of an essentially human activity one may never even "see" some of the attributed to vary in the first place. Elsewhere (Brown, 1971, 1974, 1975, 1981) I have shown how it is that use of poetic devices such as metaphor and imagery, and such human qualities as finding surprise and flipping figure and ground frequently account for our ability to see what it is that is supposedly staring us in the face all along.

References

Brown, Stephen I. "Rationality, Irrationality and Surprise," Mathematics Teaching, No. 55, Summer 1971, pp. 13-19.

Brown, Stephen I. "Mathematics and Humanistic Themes: Sum Considerations," Educational Theory, Vol. 23, No. 3, 1973, pp. 191-214.

Brown, Stephen I. "Musing on Multiplication," Mathematics Teaching, No. 61, December 1974, pp. 26-30.

Brown, Stephen I. "A New Multiplication Algorithm: On the Complexity of Simplicity," The Arithmetic Teacher, Vol. 22, No. 7, November 1975, pp. 546-554.

Brown, Stephen I. "From the Golden Rectangle and Fibonacci to Pedagogy and Problem Posing," Mathematics Teacher, Vol. 69, No. 3, March 1976, pp. 180-186.

Brown, Stephen I. Some "Prime" Comparisons. Reston, Virginia: National Council of Teachers of Mathematics, 1978, pp. x, 106.

Brown, Stephen I. "Some Limitations of the Structure Movement in Mathematics Education: The Meaning of 'Why'," Mathematics Gazette of Ontario, Vol. 17, No. 3, March 1979, pp. 35-40.

Brown, Stephen I. "Ye Shall Be Known by Your Generations," For The Learning of Mathematics, No. 3, March 1981, pp. 27-36.

Brown, Stephen I. "On Humanistic Alternatives in the Practice of Teacher Education," Journal of Research and Development in Education, Vol. 15, No. 4, Summer, 1982, pp. 1-12.

Brown, Stephen I. "Distributing Isomorphic Imagery," New York State Math Teachers Journal, Vol. 32, No. 1, Winter 1982a, pp. 21-30.

Brown, Stephen I. "Problem Posing: The Problem Generation Gap," Math Lab Matrix, No. 16, Fall 1981a, pp. 1-5.

Brown, Stephen I., and Walter, Marion I. The Art of Problem Posing. The Franklin Institute Press, 1983.

Dewey, John. Reconstruction in Philosophy, Henry Holt and Co., N.Y., 1920.

Dewey, John. How We Think, Heath, N.Y., 1933.

Getzels, Jacob W., and Jackson, Philip W. Creativity and Intelligence: Explorations with Gifted Students, John Wiley & Sons, N.Y., 1961.

Gilligan, Carol. In A Different Voice, Harvard University Press, Cambridge, 1982.

Higginson, William C. Toward Mathesis: A Paradigm for the Development of Humanistic Mathematics Curricula, Unpublished doctoral dissertation, The University of Alberta, 1973.

Hofstadter, Douglas. "Metamagical Themas: Variations on a Theme as the Essence of Imagination," Scientific American, October, 1982, pp. 20-29.

Kohlberg, Lawrence. "Moral Stages and Motivation: The Cognitive Developmental Approach," in Lickona, Thomas (editor), Moral Development and Behavior: Theory, Research and Social Issues, Holt, Rinehart and Winston, 1976, pp. 31-53.

Lickona, Thomas (editor). Moral Development and Behavior: Theory, Research and Social Issues, Holt, Rinehart and Winston, 1976.

Lipman, Matthew, Sharp, Ann S., Oscanyan, Frederick. Philosophy in the Classroom, Institute for the Advancement of Philosophy for Children, New Jersey, 1977.

Lyons, Nona P. "Two Perspectives: On Self, Relationships and Morality," Harvard Educational Review, Vol. 53, No. 2, May 1983, pp. 125-145.

McGinty, Robert L., and Meyerson, Lawrence, N. "Problem Solving: Look Beyond the Right Answer," Mathematics Teacher, Vol. 73, No. 7, October 1980, pp. 501-503.

Oliver, Donald, and Newman, Fred. Clarifying Public Controversy: An Approach to Teaching Social Studies, Little, Brown, 1970.

Snow, C. P. The Two Cultures: and a Second Look, The New American Library, New York, 1959.

Taba, Hilda. "The Problems in Developing Critical Thinking, Progressive Education, 1950, pp. 45-48.

Walter, Marion I., and Brown, Stephen I. "Problem Posing and Problem Solving: An Illustration of their Interdependence," Mathematics Teacher, Vol. 70, No. 1, Fall 1977, pp. 4-13.

Walter, Marion I., and Brown, Stephen I. "What If Not?" Mathematics Teaching, No. 46, Spring 1969, pp. 38-45.

Wheeler, David. "Mathematization Matters," For the Learning of Mathematics, Vol. 3, No. 1, July 1982, pp. 45-47.

WORKING GROUP A

DEVELOPING STATISTICAL THINKING

CANADIAN MATHEMATICS EDUCATION STUDY GROUP
JUNE 1983 MEETING (VANCOUVER)

Report of Working Group A

Developing Statistical Thinking for All

by Claude Gaulin and Jim Swift, Co-chairmen

The Working Group set as its aim the development of a set of guidelines for the introduction of work on statistical thinking into the core curriculum. The concentration on STATISTICAL thinking was deliberate. The group recognized the importance of also including PROBABILISTIC thinking in the core curriculum, but in the limited time available for discussion, it was judged preferable to concentrate on the area of developing statistical thinking for all.

A pre/corequisite of the development of statistical thinking is a SENSE OF NUMBER, which must be an important objective of the mathematics core curriculum. Number sense is understood to include such concepts as estimation, accuracy and size, and such skills as rounding and making approximations.

GOALS FOR THE DEVELOPMENT OF STATISTICAL THINKING FOR ALL

1. To develop critical attitudes towards conclusions based on commonly used statistical arguments.
2. To develop those skills of data exploration necessary for achieving the first goal.
3. To develop an awareness of the uses of statistical arguments.

APPROPRIATE STATISTICAL TOPICS FOR THE CORE CURRICULUM

The topics fall into two broad areas:

(A) The tools of data analysis

(a) *Numerical summaries*

Collecting data and presenting them in the form of tables, averages, percentages, proportions, etc.

Interpreting information presented in the forms mentioned above.

Using averages and measures of variability to illuminate data.

Detecting patterns in data.

Recognizing appropriate and inappropriate use of numerical summaries.

(b) *Graphical summaries*

Collecting data and presenting them in graphical form. Appropriate forms include: circle charts, scatter plots in one and two variables, box and whisker plots, stem and leaf plots, bar graphs, etc.

Interpreting information presented in the graphical forms mentioned above.

Using a variety of plots to illuminate a collection of data.

Detecting hidden patterns in data.

Investigating values that appear "different" (outliers).

Making comparisons between collections of data presented in graphical form (e.g. comparing box and whisker plots of two sets of heights).

Emphasis and methodology

The emphasis should be towards providing ways of overcoming the mistakes of judgement that so often arise when data are examined. The work of Kahneman and Tversky [1] has clearly shown some of the kinds of mistakes that can occur in this context.

Students should be encouraged to develop their own interests by collecting and examining data from any subject that attracts them. Project work, involving the planning and execution of an experiment that includes the collection and interpretation of data, is also a most worthwhile activity. Much attention could also be given to the compilation of a collection of interesting activities, data examples from newspapers, case studies from practicing statisticians, etc. that will illuminate the teaching of statistics. [2]

The computer and the calculator offer considerable opportunities for enhancing the development of statistical thinking. Students can concentrate on the use of the tools of data analysis, not as ends in themselves, but as a way of extracting information from an interesting data set. The computer also facilitates the exchange of data sets in a format that allows students a convenient form of access.

Statistical thinking is not confined to the mathematics curriculum. Teachers are strongly encouraged to look for applications of statistics in other subject areas. In addition, the development of statistical thinking involves the use of many mathematical skills and techniques, e.g. work on ratios, proportions, graphs and number sense. Such reinforcement is a positive feature of a stronger emphasis on statistical thinking.

(B) Sample surveys and their interpretation

(a) Developing an understanding of, and a critical attitude towards, statements made about surveys.

(b) Recognizing misleading interpretations of survey data. (For example, such statements as "the Liberals have increased their share of the popular vote by 2 %" are misleading when the poll is only accurate to 4 percentage points.)

(c) Developing an intuitive understanding of how a sample can give information about a population.

(d) Using elementary methods of obtaining a sample from a population.

Emphasis and methodology

The emphasis should be towards those activities that develop an INTUITIVE understanding of the uses and limitations of surveys. Methods that use technical jargon and emphasize probability models are not considered appropriate for the core curriculum. One successful approach has been to observe the variability that occurs when samples are taken from a YES/NO population and to summarize data from 100 such samples in a box and whisker plot. [3] This approach does not involve the use of probability statements connected with surveys. In its deliberations about methodology, the group was very conscious of the importance of dealing with misconceptions concerning surveys like those shown in the work of Kahneman and Tversky [1]. A valuable goal in teaching this topic is to aim at overcoming such misconceptions as occur at a very basic level.

Students should continually challenged to ASK questions and be critical about a survey. Appropriate questions might include: how were the data collected? Did every member of the population have the possibility of being included in the sample? Was the sample representative? Etc. Such questions are a valuable source of learning, particularly when applied to the surveys that are often conducted in other courses in the school curriculum. There is no better way of revealing the limitations of survey techniques than by being actively involved in one. The discussion following such involvement can be illuminating.

The area of sample surveys is one of the most frequently reported areas of statistical thinking. There should be no difficulty in including a strong emphasis on the use of newspaper sources when this topic is being taught.

[1] D. Kahneman, P. Slovic & A. Tversky Judgment under uncertainty: heuristics and biases. Cambridge University Press, 1982.

[2] Cf. Statistics Teacher Network, edited by Prof. Ann Watkins, Dept. of Maths., Los Angeles Pierce College, 6201 Winnetka Ave., Woodland Hills, CA 91371.

[3] . Schaeffer & J. Swift Information from Samples. (To be published by the ASA/NCTM Joint Committee on the Curriculum in Statistics and Probability).
Note: a YES/NO population is one to which the words YES or NO can be attached to each member of the population (example: a population of black and white balls in a sampling box).

APPENDICES

Appendix 1 Statistics in the School Curriculum, by R.J. MacKay

STATISTICS IN THE SCHOOL CURRICULUM

R.J. MacKay *
University of Waterloo

The purpose of this note is to outline the major problems of including "respectable statistics" in the school curriculum. By specifying these problems, I think we can direct our energies usefully and not waste a large amount of time discussing irrelevancies.

My first point is that there is no need to decide what the important topics are. The list given below can be generated using many different criteria, but I believe in the end that the list produced will look much like the one I'm giving. The criteria I've used to select these concepts are first that the topic should lead to the understanding of a frequently encountered concept. The topic should be "vivid and exciting" and further teachers should be able to link the topic to other parts of a mathematics/computer science curriculum.

The first criterion is one of relevancy. The second is useful in considering how a topic is to be presented. The third criterion is pragmatic, and recognizes that statistical ideas will be presented by non-specialists in a mathematics or computer science setting. For the teachers' sake, the statistical ideas must be tied to their own discipline.

The topics given below can be presented at many levels. I hope my brief examples will give you the flavour that I feel is required.

*The ideas on this note arose from discussions with Jim Swift, Roger Purves, Brian Graham, Alf Waterman, Ted Beuttery, Jim Nakamoto and others.

Potential Topics

A. Number sense

(a) softening big and small numbers "If a super computer can perform 50 million calculations per second, and 150 mathematicians can perform the same calculations in a week, how many calculations does each mathematician make per second?"

(Victoria Times-Colonist, April 24, 1983).

(b) biography of a number "22000 fewer claims needed"

(Victoria Times-Colonist, February 7, 1983).

B. Sample surveys

(a) "representative samples"

(b) from sample to population

(c) sampling variability

(d) non-sampling errors.

C. Statistical relationships and causation

(a) it looks like a pattern but it's only chance variation.

(b) it looks like chance variation but there is a pattern.

(c) making comparisions in a non-deterministic setting.

D. Data analysis tools

(a) numerical summaries (eg. averages, percentages, tables).

(b) graphical summaries (eg. scatter plots, box and whisker plot etc

Notes

(a) To make these topics vivid, there are several possibilities:

(i) the topics can be used to motivate mathematical ideas, for example using scatter plots of real data to introduce cartesian coordinates.

(ii) the topics can be introduced by way of a "story" or classroom experiment in which the questions of interest are asked first.

(iii) topics can be introduced as part of a computer science exercise.

(iv) statistical ideas can be embedded in a mathematics problem. eg. Given five observations plus an unknown sixth value x, plot the mean and the median of the data as x varies.

(b) Probability has been excluded from the list except for the notion of chance variation. I think probability can be included in the school curriculum but it should be separated from statistical ideas.

(c) Formal statistical procedures should not be included in the school curriculum. These methods are irrelevant for most students.

Problems

Here is a list of problems which I think can be addressed by this workshop using the given topics as a reference set.

1. identification of source materials (examples, potential projects, classroom experiments, test/homework problems, computer software etc.)

2. development of a delivery system to make these materials available to practising teachers

3. development of source materials with Canadian content (or better, of interest to Canadian students)

4. creation of ideas for linking statistical concepts to mathematics and computer science curricula.

5. teacher training/re-training.

WORKING GROUP B

TRAINING IN DIAGNOSIS AND REMEDIATION FOR TEACHERS

REPORT TO CANADIAN MATHEMATICS EDUCATION STUDY GROUP

Re: Working Group B: Training in diagnosis and remediation for teachers.
June 8 to 12, 1983

In recent years there has been considerable emphasis on diagnostic-prescriptive teaching. In mathematics education this has taken the form of analyzing computational strategies used by children and then recommending appropriate reteaching of incorrect procedures. Associated with this concern, a few educators have prepared mathematics diagnostic inventories useful in the assessment process. Further, several institutions have established mathematics clinics. Some clinics offer a diagnostic service only; others offer an associated remedial component. These clinics are usually justified to the parent institution in that they provide a service to the community, offer a research setting for their faculty, and can be used to train teachers. Occasionally a course is offered in "diagnosis and remediation in mathematics" utilizing the clinic facilities. Typically these courses are available to graduates or senior undergraduates. A student who elects to take only the regular introductory mathematics methods course would likely have very little exposure to the "diagnostic and remedial" concepts.

The purpose of this Working Group is to examine more closely a recommendation that all teachers be trained to handle diagnosis and remediation in the regular classroom and what form this might take. To facilitate discussion, selected members would be asked to prepare papers which summarize diagnostic and clinic models, both for preservice and inservice situations, and raise questions which focus on this training.

REPORT OF WORKING GROUP B: Training in diagnosis and remediation for teachers.

LEADERS: Douglas R.M. Edge, University of British Columbia
David F. Robitaille, University of British Columbia

In the past several years working groups of the Canadian Mathematics Educators Study Group have concentrated on various aspects of diagnosis and remediation. This year the working group focused on what training, if any, teachers should be given to develop their mathematics-related diagnostic and remedial abilities. Discussion was divided into three areas:

- regular education undergraduate students
- special education
- graduate students and research

The outline of this report parallels that division. Key points are noted along with resulting suggestions and recommendations.

1. Regular undergraduate teacher preparation:

a) It was acknowledged that many student teachers have great difficulty with elementary mathematics concepts. Although several approaches to this problem were considered, one was presented in some detail. All students enrolled in an elementary mathematics methods course were tested for functional numeracy on content typically found in grade seven and eight mathematics textbooks. Students who did not achieve a particular score were then asked to report to a diagnostic centre for further testing. These students were then directed toward appropriate reference material. No specific remediation was provided.

Several issues were raised which result in the need for further study. What constitutes functional numeracy? How can it be measured? What role should the university play in providing the needed remediation?

b) When student teachers are introduced to the concept of diagnosis and remediation it appears that the mathematics content is heavily biased in favour of arithmetic. Diagnosis rarely involves geometry, problem solving or estimation. Remediation tends to emphasize basic facts, place value and algorithms. Several educators maintained that the schools had obligations to teacher all children certain "basics" including the basic facts and relevant algorithms. Others noted that with the availability of hand held calculators the extent of this obligation had to be reevaluated.

c) It was noted that until recently in special education, diagnosis typically concentrated on identifying perceptual deficits, such as auditory or visual perception dysfunctions, of non-achieving students. In mathematics education, the diagnosis was much more task-analytic oriented. That is, there was more of an attempt to identify specific content objectives where remedial work was needed.

It was recognized that the current trend toward the integration of both approaches is desirable. Further study is needed on the extent of the integration and on the emphasis the perceptual deficit component should have in a mathematics methods course.

d) In terms of what to expect in a regular mathematics methods course, the point was made that as good teaching would routinely involve diagnosing students' strengths and weaknesses, it is difficult to differentiate between "diagnostic-remedial methodology" and "traditional classroom methodology". It was suggested that a methodology course would include the teaching of certain principles such as:

- introduce new concepts using materials then move toward a symbolic level
- use a four-step procedure to illustrate how to appropriately transfer from the use of concrete devices to a symbolic level; and
- utilize a teaching model that incorporates exploratory, understanding, consolidating and problem solving levels.

It was recommended that this introductory methods course not concentrate on formal diagnosis, such as the use of standardized tests, but rather on the preparation and uses of informal tests. Although there was some concern expressed about the value of error pattern analysis due to an inability to classify and describe all errors, there was a strong feeling that student teachers should be exposed to the concept. It would serve to sensitize the student teachers to the kinds of errors children can make and as a result would help them bring a preventative focus to their teaching.

2. Special education undergraduate training:

a) Of the several programs discussed, it appears that special education students are required to take courses that fall broadly into three categories. Firstly, they take some courses that are in common with regular education students such as reading and mathematics methods. Secondly, they take special education-specific courses such as introduction to the mentally retarded or the learning disabled. These courses may or may not include components related to testing (standardized or informal), to developing diagnostic skills and to teaching/reteaching techniques. Thirdly, special education students generally select courses from an optional list which, although varying greatly from institution to institution, often does include a course in diagnosis and remediation of mathematics learning problems.

In this latter regard, that is the aspect of an optional diagnosis and remediation course in mathematics learning, some concern was expressed in that many graduates of these programs will work as resource personnel in learning assistance or resource room centres. Given that, in addition to reading and language arts, much of the work of a learning assistance centre is focussed on mathematics, a course in corrective mathematics techniques should be viewed as compulsory to the training of the specialist. It was noted that unless these teachers are given training in diagnosis and remediation they are likely to utilize such standardized tests as the Wide Range Achievement Test (WRAT) or the KeyMath and make specific instructional recommendations in spite of the survey nature of the tests.

Several members of the group also expressed the opinion that the training of special education teachers at the undergraduate level should be reviewed. These members indicated that the skills required of a competent remedial teacher could be obtained only by first working with children in regular classroom settings for several years and then by taking specialized training in diagnostic-prescriptive techniques, likely at a graduate level.

b) Clinical programs were reported to have varying degrees of success. Some clinics are associated with the teaching of a particular course. Although this provides some hands-on experience, difficulties with long range planning, following-up remediation, biased referring population, and travelling time limit practical experience for the student teacher. Other clinics are operated independent of course teaching. In this case the remediation does not have to be tied to the semester system and it does permit working with a child for a longer period but it involves other problems such as the paying and supervising of qualified tutors.

Clinics operating during the summer session share some of the same features as clinics held during the fall or winter but they have two major advantages. Children can spend more hours per day studying mathematics. And, children are more likely to participate in instructional sessions on problem solving, measurement and geometry.

No specific recommendations or suggestions were forthcoming.

c) The issue of training special education teachers for the secondary level was briefly discussed. There do not seem to be any currently existing in Canada. The situation is critical and requires immediate attention. Needs were identified for the development of an appropriate content taxonomy as well as for useful diagnostic instruments. Further information is also needed on remedial techniques. Are techniques that are routinely recommended for use with elementary school children applicable to students in secondary school? Should instruction for meaning always precede drill? Which topics must be taught initially using concrete materials?

3. Graduate programs and research:

a) A proposal for a Master of Education degree program in mathematics education with a focus on diagnosis and remediation was presented. This proposal, presented in Appendix A, suggested that courses in mathematics education (foundations, advanced methods), educational foundations (learning theory, human development), research and measurement, and field experience all be required components. It was also suggested that students be given opportunities to select from a series of optional courses which would include mathematics, special education, and reading education.

Although there appeared to be general support for the proposal, two additional recommendations were made. Firstly, a course in standardized testing should be required. And secondly, some attention should be given to the preparation of an integrated Masters of Education program in diagnosis and remediation. For example, joint programs in mathematics and reading, mathematics and early childhood or mathematics and special education should be proposed and implemented.

b) Conducting research in diagnostic-remedial settings has presented special problems. It has been difficult to carry out experimental studies using large sample sizes given that clinical programs often contain too few subjects. Hence, researchers tend to rely on methods which use case study approaches. One concern related to the use of case study research was expressed. This research tends to be hypothesis - generating in nature rather than leading to specific or "practioner-oriented" conclusions. But, it was noted that currently more research is utilizing case study methodology. This is likely due to the respect accorded to [illegible] high quality, insightful reporting based on these [illegible].

Rather than make specific suggestions or recommendations relating to diagnostic-remedial research, the group focussed on past and current studies. Error pattern analysis was discussed. Results are often difficult to classify and this led to some consideration of the value of pursuing this line of study.

Clinics were seen as good settings to examine various pedagogical models and remedial techniques: models such as teaching cycles or transferring from concrete to symbol, and remedial techniques such as the use of contrasting algorithms or of specific manipulative materials. The extent of computer use in remediation was mentioned and appears to require much further investigation.

An additional area that received consideration was the development of diagnostic instruments. Examples given included the U.B.C. MEDIC Checklist and the Maryland Diagnostic [illegible] Test. Carryl Koe (U.B.C.) discussed her current re[illegible] involves the development of a diagnostic checklist for solving linear algebraic equations.

SUMMARY

Throughout the several days of discussion several major themes appeared to recur. Firstly, diagnosis is an essential component of all good teaching [illegible] hence must not only be considered for that group of children identified as in need of special help. Consequently all pre-service teachers need training in this area. This training should include experience in the preparation and interpretation of informal testing procedures and with the development and implementation of appropriate remedial techniques.

Secondly, much more extensive use must be made of the hand held calculator. This is imperative when working with older children who may have been struggling unsuccessfully for several years with particular facts and algorithms. It was not as clear that the computer should receive the same emphasis. More study of the role of the computer in diagnosis and prescription is needed.

Thirdly, special education students require much more expertise related to diagnosis and remediation of mathematics learning. When these students become teachers they are typically expected to have a high level of ability in working with children who have mathematics learning difficulties yet it appears that, at best, courses which would prepare them for this task are optional.

Other themes were also apparent. There is no intention to suggest that these are less important. It is simply that they received less attention than the first three themes noted above.

Firstly, Master's level graduate degree programs need be prepared. Courses should maintain a focus on mathematics learning, but need to address diagnostic and remedial concerns in related fields, such as reading and special eudcation. It is also possible that this kind of graduate program should come to be seen as the way to train "special educators", rather than undergraduate degree programs.

Secondly, vastly more interest needs to be shown toward the learning problems of adolescents at the secondary school level. The direction for this interest certainly must be directed toward diagnostic and remedial issues but must also include the development of relevant, application-oriented courses. Watered-down versions of existing courses must be avoided.

The working group addressed numerous issues. In some cases the making of specific suggestions seemed appropriate. In other cases, it was recognized that further study of a topic was required before recommendations should be made. It is hoped that future working groups would be organized to examine in some detail one or more of these issues.

APPENDIX A

PROPOSAL FOR A MATHEMATICS EDUCATION PROGRAM FOCUSSING ON DIAGNOSIS AND REMEDIATION IN MATHEMATICS.

PREREQUISITE: Mathematics methods and content (369 or 404)*
Introduction to Diagnosis and Remediation (471)

SUGGESTED COURSES:	UNITS
1. Math Ed: Diagnosis and Remediation (56[illegible])	3.0
Foundations (545)	1.5
Math Teaching: Elementary (547) or Math Teaching: Secondary (548)	1.[illegible]
2. Foundations: Learning Theory (501)	1.5
Human Development (505)	1.5
3. Research and Measurement: Research in Education Methodology (508)	1.5
4. Practicum: Field Experience (598)	3.0
OPTIONAL COURSES: Selected from following -	4.5
	18.0

History of Mathematics (485)
Mathematics Education: Elementary (488)
Mathematics Education: Secondary (549)
Remedial Reading (476)
Introduction to Research (481)
Introduction to Statistics (482)
Behaviour Disorders (515)
Learning Disabilities (526)
Hearing Impaired (530)
Multiple Handicapped (537)
Early Childhood
Testing
Adult Education
Education for the Gifted

* Numbers refer to U.B.C. course numbers.

WORKING GROUP C

MATHEMATICS AND LANGUAGE

Mathematics and Language

> I am impressed by the inadequacy of language to express our conscious thought, and by the inadequacy of our conscious thought to express our subconscious. The curse of philosophy has been the supposition that language is an exact medium. Philosophers verbalise and then suppose that the idea is stated for all time. Even if it were stated, it would need to be restated for every century, perhaps every generation. Plato is the only one who knew better and did not fall into this trap. When ordinary methods failed him, he gave us a myth, which does not challenge exactitude but excites reverie. Mathematics is more nearly precise, and comes nearer to the truth. In a thousand years it may be as commonly used as is speech today.
>
> – A.N. Whitehead

> What is a satisfactory definition? For the philosopher or the scholar, a definition is satisfactory if it applies to those and only those things that are being defined; this is what logic demands. But in teaching, this will not do; a definition is satisfactory only if the students understand it.
>
> – H. Poincaré

> There is no more reason why a person who uses a word correctly should be able to tell what it means than there is why a planet which is moving correctly should know Kepler's laws.
>
> – B. Russell

In spite of the prefatory quotations, this is not intended to be a well-aimed, thoroughly composed paper, but only a ramble through some small parts of the territory. The hope is that the remarks may trigger thoughts that are worth pursuing. The area is vast and mainly unexplored.

I shall keep well away from the region signposted Mathematics is a Language. I believe it to be uninhabited.

Two papers which attempt some coverage of the ground for the mathematics educator are Aiken (1972) and Austin and Howson (1979). Neither seems to me to be very satisfactory, though the second is much more useful than the first. Both suggest that comprehensive surveys are really out of the question – there is too much to cover. More practical, and more useful, would be a map (graph, flow-chart, diagram) of the whole area supplemented by surveys of particular localities.

For this paper I consider three chunks.

__Mathematics as an outcome of language__

When we teach a language (native or foreign) we usually assume that the students already have experiences that they can talk and write about in the language. When we teach mathematics we usually assume that we have to teach the language of mathematics and, simultaneously, what the students may say and write in it. (Sinclair, 1980)

But the students learning mathematical language already possess functional, and essentially grammatical, speech; that is, they can, without knowing what they are called, operate correctly with the various "parts of speech". Each part of speech is associated with certain awarenesses that appear to be mathematical. For example:

Knowing how to use ...	requires the awareness of ...
nouns, verbs	classes (of objects, actions)
adjectives, adverbs	intersection/inclusion of classes
pronouns	substitution
prepositions	relation
tenses	transformation

Again, anyone who can use the following statements equivalently has a grasp of an "algebra" of language:

Je donne mon canif à elle	Il donne son canif à moi
Je lui donne mon canif	Il me donne son canif
Je le lui donne*	Il me le donne*

(* and is aware that the algebra may occasionally be disturbed by other demands - for euphony, for instance).

So in a certain sense some mathematical awarenesses precede (or accompany) the acquisition of speech. How, then, does our assumption arise that mathematics can only follow an induction into its special language?

We might obtain some further illumination by looking at "pathological" cases. Can deaf children, without normal speech, learn mathematics? (A short classic paper by Wertheimer and studies by Hans Furth look at this.) Can children brought up in a totally different culture learn mathematics as we know it? A paper by Gay (1974) suggests that all known languages have the basic structural elements of conjunction, negation and quantification and that these are sufficient for the development of mathematics.

The evidence is overwhelming that there is more to mathematics than its special language. However stifled and stunted mathematical development would be without the language, it is the "more" that gives meaning to the words and the signs and the linguistic conventions.

I omit here the complex and fascinating matter of the relationship between mathematical and "natural" language.

Characteristics of mathematical language

Following Whitehead, we may see mathematical language as more precise than ordinary language. But there are at least two qualifications we may make. One is that even when used by mathematicians the language is not precise in the sense of "free from ambiguity". Par un abus de langage is a common bridging phrase in the writings of some French mathematicians, who care about these things. The other is that mathematical language is not precise in the sense of "being clear". We have only to consider legal language to see that the claims of clarity and exactness are not always compatible. (Teachers seem more prone than lawyers to confuse the two qualities.)

How do we learn, say, the word "rectangle"? To give it an initial meaning we distinguish the object described from objects described by "square", "parallelogram", etc. But in order to give a definition, one of the properties is dropped ("two long sides and two short sides"). This enables us to say, for example,

"a square is a rectangle"
"a rectangle is a parallelogram"

These statements have the same formal structure as "a rose is a flower" (the class of roses is included in the class of flowers). But we do not first encounter the word "flower" by distinguishing the thing it describes from the thing described by "rose", or vice-versa.

We write, say, $2 < 5$ and speak "two is less than five". Is 2 less than 5 because the former is the cardinal number of a proper subset of the set of which the latter is the cardinal number? Or because 2 occurs before 5 in the counting order? If we substitute the criterion of a positive difference ($a < b$ if $b - a > 0$) we find ourselves writing $-5 < 2$, for instance. Is -5 less than 2? In either of the above senses? See Pimm's paper (1980) to see how words and symbols defined in certain situation are applied as metaphors in different situations... "Morphisms preserve structure but do not preserve meaning." Structure-equivalents are not necessarily meaning-equivalents.

In an interesting paper, Kaput (1979) takes two "axioms" - mathematics is a formal structure, and mathematics is based on experience. He suggests that mathematical notation lies on the border of the two zones. Notation which cannot be related in some way to experience is (literally) nonsense. But formal mathematics is atemporal whereas experience is always in time. So to tie notation to experience is to see it arising from a process (in time) whereas as a feature of the formal structure it is also a product (timeless). In his terms, to make sense of mathematical notation we must "anthropomorphise it".

Mathematical language has a history - it is a sort of historical sediment. Knowing the history may at times illuminate, at others exasperate. (Why, oh why, should we have to bear with the clumsiness of the radical sign, and why cannot we write our numerals so that the place value increases from left to right?)

The power of mathematical language is undeniable, but perfect it is not.

Language use in the mathematics classroom

A number of assorted aspects occur to me.

What proportions of classroom time are spent (by the teacher) in the four modes of listening, speaking, reading and writing? How do these compare with other subjects, or with the work of a (non-teaching) mathematician? Why is there so little class or group discussion in mathematics classrooms? Whatever one's views on the discussability of mathematics per se, there is always the possibility of discussing alternative solutions to a problem, variations in algorithms, contrasting proofs.

But discussion has other uses. It is perhaps the principal method we have for negotiating the meanings of words and the validity of arguments. A good discussion is, for the participants, a process of "becoming clear.". Lakatos (1975) casts his thesis in the form of a guided discussion to show that mathematics, too, requires negotiation and an evolving clarity.

Definitions are not devices for making things clear. They enable thought and argument to be exact, but that is not (I think) their principal function. A definition flags an idea, a perception, an awareness, and says of it that it has a future, that the idea will be productive. The form of a definition may vary, as we know, to satisfy criteria of convenience or elegance, but these are subordinate to the criterion of significance. We tend to think of an organised part of mathematics as a collection of theorems, but it is the definitions that power the development. Unfortunately very few classrooms give students even a glimpse of this process.

Why are "word problems" such a bugbear for mathematics students and teachers? In other subjects all the problems are word problems! Part of the difficulty lies in the fact that students (Lorenz, 1980) and teachers (Nesher and Taubel, 1975) are all the time colluding to convert word problems into algorithmic exercises.

Lorenz also points out (quoting Bauersfeld) that in no other subject does the teacher tend to identify the answers to questions by their form rather than their content, even to the extent of rejecting correct answers not given in the prescribed form. Questioning is not, in any case, a skill that mathematics teachers are generally good at. (I have no hard evidence for this.) It is too easy to formulate apparently straightforward questions, perhaps. I think that to ask "What do we mean by fraction?" is as difficult to answer as "What do we mean by justice?" Teachers, and mathematics teachers in particular, tend to forget that Socrates spent most of his life demonstrating that what we incontrovertibly know we cannot necessarily tell.

David Wheeler
May 26, 1983

References

L.R. Aiken, *Language factors in learning mathematics.* Mathematics Education Reports, Oct. 1972

J.L. Austin and A.G. Howson, "Language and mathematical education". *Educational Studies in Mathematics* 10, 2 (May 1979):161-177

J. Gay, *Mathematics, language and effective learning.* UNESCO, 1974

J. Kaput, "Mathematics and learning: roots of epistemological status". In: J. Lochhead and J. Clement (eds.) *Cognitive process instruction.* Franklin Institute Press, 1979

I. Lakatos, *Proofs and refutations.* Cambridge University Press, 1976

J.C. Lorenz, "Teacher-student interactions in the mathematics classroom: a review". *For the Learning of Mathematics* 1,2: 14-19

P. Nesher and E. Teubal, "Verbal cues as an interfering factor in verbal problem solving". *Educational Studies in Mathematics* 6, 1 (July 1975): 41-51

H. Otte, "Textual strategies". *For the Learning of Mathematics* 3, 3:15-28

D. Pimm, "Metaphor and analogy in mathematics". *For the Learning of Mathematics* 1, 3: 47-50

H. Sinclair, Language and mathematics in acquisition. Address to ICME-IV Congress, Berkeley, 1980

M. Wertheimer, *Productive thinking.* Harper, 1959

Language Usage in Mathematics

Mathematics educators and others versed in the use of sophisticated mathematics are sometimes unaware of the difficulties learners might encounter as a result of the language used in the presentation and discussion of mathematics. Inaccurate, inappropriate, or ineffective use of language is often a contributing factor in many learners' conception of mathematics as "something you do, but not something you understand."

During the past several months, by attempting to listen carefully to myself and to my students, I have found that difficulties in language usage in mathematics seem to fall into one of the following categories:

I. Situations where language is constrained by standard usage.

II. Situations where better alternatives are available.

III. Situations exemplified by careless or incorrect usage.

An important example from the first of these categories is the naming of the "teens". Irregularities in the words used, the word order, and the spellings are all present. In this situation there appears to be no immediate remedy except for the teacher to be aware of the inherent difficulties and to be sensitive to problems they may cause in the classroom. Some other examples from this category are:

1. "Reduced" fractions: Does reducing a fraction make it smaller?

2. "Like" terms: $2xy$ and $3xy$ are "like" terms.
   $2\sqrt{xy}$ and $3\sqrt{xy}$ are "like" terms.
   $\sqrt{2}xy$ and $\sqrt{3}xy$ are not "like" terms.

3. "Variable": How variable is the "variable" in the equation $2x+1=5$?

4. In writing an expression for "the square root of seven more than a number", should one write $\sqrt{7}+x$ or $\sqrt{7+x}$?

The second category, situations where better alternatives are available, appears to be the richest, both from the viewpoint of the number of examples available, and the decisive role that the teacher can play in alleviating the difficulty. An elementary level example of this type occurs in the language used to introduce multiplication and division. "Two threes are six" and "How many threes in six?" seem preferable to "Two times three is six" and "How much is six divided by three?" for two important reasons.

First, the former use language already available to the students, and second, they do not obscure the relationship between the operations. Some other examples in this category are:

1. Names of decimals: .17 should be read "seventeen hundredths" rather than "point seventeen". The latter obscures meaning and makes other connections more difficult.

2. Names of polynomials: x^2 should be read "x to the second power", not "x squared", unless a model for the latter has already been developed. A similar argument holds for x^3.

3. Classification of equations: "First degree, second degree, third degree" is probably a more meaningful sequence than "linear, quadratic, cubic".

4. Subtraction: Using "difference" rather than "take away" is suggested by certain manipulative models (e.g. Cuisenaire rods) and more easily leads to transformation useful in rapid computation.

5. Bases of numeration: Using the traditional language when working in other bases of numeration is cumbersome and obscures algebraic relationships. For a further discussion of this topic, see the 1980 (Laval) Proceedings of CMESG. Also refer to Trivett and Gattegno (see bibliography).

The third category is characterized by careless or incorrect language usage, and can be eliminated by the careful teacher. Some examples from this category are:

1. Names of fractions: ¼ should be read "one fourth", not "one over four". The latter is devoid of meaning and can lead to serious problems in computations with fractions.

2. Reading and writing large numbers: Place value names are of little use in learning to read and write large numbers. A more effective approach is to "read" the commas (Gattegno).

3. Time and numbers: "One twenty five" is a reading of 1:25 and refers to time. "One hundred twenty five" is a reading of the number 125.

I do not take the preceeding categorizations to be definitive or the lists of examples to be exhaustive, however I believe that this or a similar exercise can be of value to classroom teachers and teacher trainers. It has led me to consider the following criteria for making choices about what language to use:

1. Use language that emphasizes, or at least does not obscure, mathematical structures.

2. Use "available" language, i.e., language brought to the situation by the students or suggested by manipulative models in use.

3. Choose clarity over precision.

4. Use language in a consistent manner.

Martin Hoffman 7/83

Bibliography

Bell, F.H. "Posing and Solving Verbal Problems". Mathematics Teacher, Dec., 1980.

Confrey, Jere. "Concepts, Processes, and Mathematics Instruction". For the Learning of Mathematics, 1981, vol.2, no.1, 8-12.

Cunningham, Trudy. "Mark My Word". Mathematics Teacher, Sept., 1982.

Dawson, A.J. "Words Triggered by Images: Images Triggered by Words". Mathematics Teaching, vol.98, Mar., 1982.

Gattegno, Caleb. The Common Sense of Teaching Mathematics. New York: Educational Solutions, 1974.

Gattegno, G. and Hoffman, M. Handbook of Activities for the Teaching of Mathematics at the Elementary School. New York: Human Education, Inc., 1976.

Goutard, M. Mathematics and Children. Reading: Educational Explorers, 1964.

Henrichs and Sisson. "Mathematics and the Reading Process". Mathematics Teacher, Apr., 1980.

Herscovics and Kieran. "Constructing Meaning for the Concept of Equation". Mathematics Teacher, Nov., 1980.

Higginson, William. ""X" Arguments for the Consideration of Language by Researchers in Mathematics Education". Proceedings of the 1981 (Edmonton) meeting of CMESG/GCEDM.

Higginson, William. "Mathematizing Frogs". Mathematics Teacher, Oct., 1981.

Kline, Morris. Why Johnny Can't Add. New York: Random House, 1961.

Morley, Arthur. "Teaching and Learning Algorithms". For the Learning of Mathematics, 1981, vol.2, no.2, 50-51.

Nesher, Pearla. "The Stereotyped Nature of School Word Problems". For the Learning of Mathematics, 1980, vol.1, no.1, 41-48.

Perlman, G. "Making Mathematical Notation More Meaningful". Mathematics Teacher, Sept., 1982.

Powell, A. and Hoffman, M. "Use of Gattegno's Number Array". Proceedings of the 1980 (Laval) meeting of CMESG/GCEDM.

Resnick, P. "Some Misconceptions Concerning the Concept of Variable". Mathematics Teacher, Sept., 1981.

Tahta, D. "Words, Images, and Objects". Mathematics Teaching, 1981, vol. 96, 34-37.

Trivett, John V. ...And So On; New Designs for Teaching Mathematics. Vancouver: Simon Fraser University, 1979.

Wheeler, D., "Awareness of Algebra". Mathematics Teaching, 1981, vol. 96, 29-34.

Wheeler, D., Notes on Mathematics in Primary Schools. Cambridge: The University Press, 1967.

Components of
A "Grammar" of Elementary Algebra Symbol Manipulation

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> The Study of Algebra may be pursued in three very different schools, the Practical, the Philological, or the Theoretical, according as Algebra itself is accounted an Instrument, or a Language, or a Contemplation; according as ease of operation, or symmetry of expression, or clearness of thought, (the agere, the fari, or the sapere,) is eminently prized and sought for.
> (William Rowan Hamilton, 1837)

That mathematics can be regarded as a language has been noted by many authors (see Aiken 1972 for references). More recently, educators have begun to question either the validity, or more probably the utility, of that connection:

> It has frequently been pointed out that mathematics itself is a formalized language and it has been suggested that it should be taught as such ....
> Such statements possess a degree of validity, but would appear to be somewhat dangerous and potentially confusing. Mathematics is not a language - a means of communication - but an activity and a treasure house of knowledge acquired over many centuries. (Austin & Howson, 1979, p.176)

Indeed several of the previous speakers of this working group have expressed a similar point of view.

Attempts in the past at drawing the connection have generally focussed on identifying structures of language in mathematics or vice versa. In my current research, mathematics is regarded as a language not so that knowledge of language may be applied to mathematics, but rather so that the techniques designed for the study of language may be applied the the study of mathematics. Indeed, in the narrow sense by which linguists define the term, mathematics is unequivocally a language:

> From now on I will consider a language to be a set (finite or infinite) of sentences, each finite in length and constructed out of a finite set of elements. All natural languages in their spoken or written form are languages in this sense, since each natural language has a finite number of phonemes.... Similarly, the set of 'sentences' of some formalized system of mathematics can be considered a language. (Chomsky, 1957, p.2)

It is not possible for me, in the short time available today, to outline in any detail the linguists' methods, or to elaborate on the way in which I have adapted those methods for the study of algebra. A few words, however, are needed to define exactly what is meant by a "sentence" of elementary algebra and to identify the basic elements of which sentences are comprised.

I interpret the term sentence in linguistics to refer to the smallest unit of discourse which will normally be uttered by a speaker who is being attended by listeners. In natural language study, then, sentences correspond to the statements, questions and exclamations of normal speech. By analogy, sentences of algebraic manipulation refers to equation solving (or system of equations solving) and to the simplification of algebraic expressions. These are the minimal acts which will be accomplished by one who has embarked upon the <u>manipulation</u> of algebraic symbols. The basic elements of which these sentences are comprised are the usual algebraic symbols a,b,c,x,y,z,1,2,3,....,+,-,(,),=,etc.

The linguistic program is the development of a <u>grammar</u> which can be loosely described as a set of rules which formally operate upon the basic elements to produce the sentences of the language. In my study, the simplification of algebraic expressions has been selected as the subset of sentences of interest.

A grammar, as a mechanism for the production of sentences, can be regarded as a cognitive theory. The devices employed by the grammar are postulated to be the same ones employed by the competent manipulator of algebraic symbols. Alternative grammars may be formulated representing alternative cognitive theories. Psycholinguistics offers various paradigms for the selection of a grammar from amongst alternatives on the basis of competing psychological claims.

In the case of natural language, linguistics involves analysis at a variety of levels (phonemic, morphemic, phrase structure, semantic, etc). A grammar of algebraic manipulation likewise involves various levels of analysis. These will be outlined during the remainder of my talk.

1. Classification of Basic Elements. The first stage is the identification of the basic elements and their assignment to various classes (eg. operators, quantity symbols, and grouping symbols). As an example, the $\sqrt{\ }$ symbol is interpreted as a conjunction of two symbols: $\surd$, and $\overline{\ }$, belonging to the operator and grouping symbol categories respectively.

2. Expressions. Having established the basic elements of the theory and their classification, the next step is to rigorously define which strings of symbols will be considered as "algebraic expressions" (eg. $5x(x + y^{-2})^3$, not $5x(y^{3-)}\ )$.

3. Parsing. This component determines the parse of well formed expressions. For example, it is necessary to define $3x^2$ as representing $3(x^2)$ rather than $(3x)^2$.

4. Transformations. The fourth stage is concerned with the properties of real numbers which are used in the generation of one algebraic expression from another. These include arithmetic transformations, the standard "field axioms" of algebra, and any other number properties which a competent manipulator of algebraic symbols may bring to bear on one expression in the derivation of another.

5. Application Component. This penultimate level deals with the actual application of real number properties to syntactically determined expressions. This component specifies a decision procedure to determine whether a particular transformation is applicable to a given algebraic expression.

6. Semantic Component. Thus far the levels of analysis defined allow for the production of strings such as $4x^2 - 12x - 16 = 4(x^2 - 3x - 4) = 4(x^{1+1} - y + y - 3x - 5 + 1)$ as well as $4x^2 - 12x - 16 = 4(x^2 - 3x - 4) = 4(x - 4)(x + 1)$. Both of these involve the correct application of correct real number properties. It is necessary, however, to exclude sentences of the former sort which are in some sense algebraically "meaningless". The semantic component consists of a classification of sentence types according to the purposes which are normally associated with expression simplification such as factoring, reducing fractional expressions, rationalizing radical denominators, etc. In each of these cases it is necessary to delineate the initial configurations required and the sequence of transformations to be applied. (I have not yet constructed the semantic component).

It is postulated that these six components of the algebraic grammar represent the areas of skill and knowledge required for successful manipulation of algebraic expressions. To the extent that this is true (and to the extent that the rules specified within each component correspond to cognitive structures) many important implications to algebraic curriculum may be derived. To the extent that these claims are false, the linguistic paradigm, in algebra as in natural language, challenges researchers to devise a more adequate grammar.

References

Aiken, L. R. Lanaguage factors in learning mathematics. Mathematics Education Reports, October 1972. (ERIC Document Reproduction Service No. ED 068 340)

Austin, J. L., & Howson, A. G. Language and mathematical education. *Educational Studies in Mathematics*, 1979, *10*, 161-197.

Chomsky, Noam. *Syntactic structures*. The Hague: Mouton & Co., 1957.

Hamilton, W. R. Theory of conjugate functions, or algebraic couples ; with a preliminary and elementary essay on algebra as the science of pure time. *Transactions of the Royal Irish Academy*, 1837, *XVII*, 293-422.

WORKING GROUP D

THE INFLUENCE OF COMPUTER SCIENCE ON THE UNDERGRADUATE MATHEMATICS CURRICULUM

WORKING GROUP D

The Influence of Computer Science on Undergraduate Mathematics Education

John Poland

List of participants: Bernard R. HODGSON (Laval)
John POLAND (Carleton)
Yvan ROUX (U.Q. a Rimouski)

This working group was a continuation of one on the same topic the year before, and had been advertised as having the aim of producing working documents for publication on this topic using the fertile production of the working group in 1982. Yvan Roux attended only the first meeting where he raised the general question of what changes influenced by computer science would be appropriate in undergraduate mathematics education to meet society's needs, and after the discussion he preferred to spend his time at the meeting reading the background material to be in a better position to participate in concrete changes he anticipates in his own department in the coming year. The second meeting was a discussion of the issues with Peter Hilton. The final meeting produced a draft for the article in the appendix.

REVAMPING THE MATHEMATICS CURRICULUM:

THE INFLUENCE OF COMPUTERS

by Bernard R. Hodgson and John Poland++

Almost every mathematics department in Canada has experienced a drop in the number of students graduating with a mathematics degree at the bachelor's level; in many cases, to an unhealthy level. This phenomenon has occurred in many other countries too, and it is clear that the attractiveness of a career in our sister subject, computing, is a major factor. Computing is the new, challenging and prestigious frontier. But there are a number of key factors in this computer revolution that we feel will compel specific changes in undergraduate mathematics education. Let us spell out what we see as these key factors, the problems to which they give rise and scenarios of probable reactions and solutions.

Most important, in the next few years we can expect to see large numbers of freshmen in our mathematics classes with a substantial experience with microcomputers and their programming packages. Many provinces are committed to extensive distribution of these facilities to secondary schools and many students are eager to learn. At the undergraduate level we will see more disciplines using increasingly sophisticated computer techniques

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** This article is the outcome of a working group of the Canadian Mathematics Education Study Group (CMESG) meeting in June, 1983, a meeting made possible through a SSHRC grant. We express our indebtedness to the lively contributions of the members of the 1982 CMESG meeting on this topic too.

and backup mathematics. Of course, computer programs will continue to grow in their ability to do arduous multiprecision calculations and carry out our standard numerical algorithms (like Simpson's rule or row reduction of matrices), as well as grow in the ability to do routine algebraic manipulations (like techniques of indefinite integration or solving equations for specified variables). And the increasing ability of computer programs to carry out routine undergraduate mathematics also comes with a growth of the new area of modern applied mathematics: mathematical computer science (from computational complexity and probabilistic algorithms to formal languages and cryptanalysis).

Does mathematics as we teach it now really address these changes? We feel that most of the undergraduate introductory mathematics courses in calculus, linear algebra and abstract algebra are presented in the classroom as though computers do not exist. How can we expect to be considered as teaching to our students when for example we present the traditional techniques of integration (e.g. partial fractions) and our students know that already there are packages to do these symbolic algebraic manipulations on the computer, and in any case computer programs exist to evaluate definite integrals without using anti-derivatives? This illustrates that some of the content of these courses needs to be deemphasized, especially as it relates to the actual passage to and evaluation of solutions that computers can obtain (c.f. P.J. Hilton in [CMESG 83]). But the more we use computers for these processes, the more we will need to emphasize checking and validation. The question is that thorny one of relevance. How relevant is our approach to the calculus or algebra? How relevant is the actual content of our courses? Are there other topics we should be introducing to the students? And how relevant does mathematics seem to them as a way of solving questions with which they are or expect to be concerned? What we wish most to share here is our feeling that the attitudes and expectations of the majority of our freshmen who have some interest in mathematics is and will continue to be for some time that the most challenging and meaningful problems have to do with computers. And this must be acknowledged in our methods of motivating our students, and students from other disciplines taking our courses.

In what reasonable ways might we modify the content and style of our undergraduate mathematics teaching? It seems useful to point out that this situation can be addressed at different levels. Right in the classroom we can make use of handheld calculators or a microcomputer with a number of display units to painlessly collect empirical data as grounds for hypotheses and as a source of problems, or simply as a means of easily and effectively illustrating results. Outside the classroom, assignments to the students can involve similar computer-related methods and can incorporate experience with existing computer packages, such as LINPACK in linear algebra. Here we see the computer as a very powerful tool. Next, as we have argued above, the existence of these computer programs allows us to shift our viewpoint when we come to teach various methods of calculation. Approximation, estimation and optimization will gain in emphasis (including at the secondary school level). Algorithms are central to computing. We can expect an algorithmic way of thinking to grow in mathematics. It will stress recursion, iteration and induction as its tools, and routinely include such topics as computational effort when an algorithm is introduced, including the necessity of formalizing algorithms in order to analyze them. To meet this perspective, we could use more algorithmic, constructive methods of proof where appropriate. At the same time, we should not forget the appropriateness of many areas of mathematics to the study of exactly these new aspects, which brings us to the next level of possible modification.

For the content of our courses, the demand for an introduction to the material of mathematical compüter science is clear. The more advanced of these, on discrete structures or the design and analysis of algorithms or finite automata theory, are appearing in most undergraduate mathematics calendars and their adoption is generally not problematic. But the most elementary of these, under the umbrella title of Discrete Mathematics, is currently the subject of a debate, based in the United States, on whether to offer such a course as an alternative to the calculus in the freshmen year [RALSTON 81 and FUTURE 83]. In summary of the debate, the concensus seems to be that no satisfactory textbook (and hence no satisfactory syllabus?) for such a Discrete Mathematics course yet exists; and the calculus may be a more effective vehicle for teaching mathematical maturity, by virtue of its own maturity, depth and wide applicability. Let us look at these two points.

Frequeotly the proposed curriculum for the freshmen Discrete Mathematics course is a collection of traditional mathematical results, similar to present Finite Mathematics courses: Boolean algebra, combinatorics, induction and recursion, graphs. Is this a satisfactory and relevant approach to the problem, or is it disjointed, superficial and trivial? What is our purpose with such a course: to introduce the student to a language and some elementary results useful in studying computer science? Or can we go further and show the power of mathematical thinking? Research in modern applied mathematics shows us the relevance to the discussion and solution of major computer science problems. Can we convey this to our students convincingly? One approach might be a course on congruences over the integers, finite fields, polynomials and coding theory. Another, on the combinatorial analysis of algorithms, was outlined by H. Wilf in [FUTURE 83, p. 38] and is similar to Chapters 2 and 3 of the successful upper-level text [DESIGN 74] by Aho, Hopcroft and Ullman.

What is the basic perspective we should retain when considering these changes, what is our overall goal? The major recommendation of [CUPM 81] was to capture the students' interest and lead them to develop both the ability for rigorous mathematical reasoning and the ability to generalize from the particular to the abstract. In this context it should be recalled that the Science Council study of mathematical sciences in Canada [COLEMAN 76] found "almost all mathematics professors allege that their highest ambition in undergraduate teaching is to convey not specific content but rather a way of thinking", a way of thinking that even our colleagues in other disciplines consider important and wish their students to undergo when taking our courses. It is so easy when teaching specific content to forget that our subject matter, mathematics, is one of the greatest intellectual achievements of mankind. True, many introductory calculus courses are presented as mere exposition-regurgitation, but how much greater is the possibility that the original proposals of a Discrete Mathematics course degenerate into meaningless junk? Can we offer our students courses in which the power of mathematics can be demonstrated in computer science and the value of the computer in mathematics can be appreciated in its proper role?

Perhaps the most attractive option is to blend the new approach in mathematics outlined previously, with the traditional values of calculus-analysis-differential equations courses. Examples of such integiated courses can be found through our bibliography (E. Barbeau in [CMESG 82], [CRICISAM 68], P.S. Roberts in [FUTURE 83], P.J. Taylor in [CMESG 82], [WONNACOTT 77]), and we would appreciate hearing of others. Students graduating with such a modified undergraduate mathematics education would be better prepared for future changes and to use the full range of their mathematical training in their work. A model that appeals to us is that of someone beginning with a large database, taking the limit to obtain a continuous function incorporating this data, perhaps as the solution of a differential equation, and then solving a discrete approximation to this continuum formulation, for example using the finite element method and linear algebra.

Before you decide on the nature and details of the changes you would like to see in undergraduate mathematics education in your university, do read the well considered proposals of [CUPM 81], the many sources and ideas in [CMESG 82] and our annotated bibliography. If you begin with small changes in your courses, these will probably be mainly in style, and you should collect resources, including texts that incorporate this style in their presentation (e.g. [STRANG 80] in linear algebra; [WONNACOTT 77] in calculus). For larger, "curricula" changes you will need to convince your colleagues, both within your department and those in other disciplines, of the value and necessity of your suggested changes. A description of both the new content and style is necessary so that the spirit of the change is clearly perceived. Problems in coordinating your proposed changes with other departments are discussed by L.K. Barrett in [FUTURE 83].

We raised the question of the relevance of mathematics courses, taught in the traditional form, for students arriving at the university with a wide computer background, as many of our students will. We argued for the retention of presenting the mathematical way of thinking and showing the students the power of such ways of thought. We argued against simply replacing traditional courses with a scattered introduction to the language and background of mathematical computer science. We offered a number of suggestions on ways of producing more convincing introductions, and more important we suggested ways of making adjustments in existing courses to meet with the increasing use of computers by our students. Our emphasis was on the fruitfulness of the interaction between mathematics and computer science, and the reasonable modifications we can attempt in our courses so our students have a deeper, wider and more meaningful education in mathematics.

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[APPLICATIONS 79] Applications of Linear Algebra, 2nd edition, C. Rorres and H. Anton, Wiley, 1979.

Good (inexpensive) supplement to any standard sophomore-level textbook. Includes topics from a wide variety of fields (business, economics, engineering, physics, genetics, computer science, geometry, etc.). Chapters are independent and are rated according to their difficulty.

[APPLICATIONS 80] Applications for Elementary Linear Algebra, S.I. Grossman, Wadsworth, 1980.

Somewhat similar to [APPLICATIONS 79], but with less coverage.

[BIRKHOFF 72] "The impact of computers on undergraduate mathematical education in 1984", G. Birkhoff, American Mathematical Monthly 79(1972) 648-657.

Supports use of computers for study of the limit concept, rates of convergence, equation-solving, formula manipulation, etc. "New courses, to be widely taught by 1984": discrete mathematics, numerical mathematics".

[CMESC 82] Proceedings of the 1982 Annual Meeting of the Canadian Mathematics Education Study Group, ed. D.R. Drost, 1982. (Available from Educational Resources Information Center (ERIC), Ohio State University, Columbus, Ohio.)

Pages 51-95 give the report of the working group on the influence of computer science on undergraduate mathematics education, including appendices by eight of the one dozen participants. A lively, multifaceted set of articles, more positive in spirit than [FUTURE 83], with many concrete suggestions.

[CMESG 83] Proceedings of the 1983 Annual Meeting of the Canadian Mathematics Education Study Group, ed. C. Verhille, to appear, 1983.

Of particular interest is the address by P.J. Hilton on "The nature of mathematics today and implications for mathematics teaching" which gives both general principles and specific instances of what he would like to see taken out and put in to mathematics education at all levels.

[COLEMAN 76] Mathematical Sciences in Canada: Science Council of Canada Background Study 37, A.J. Coleman et al., Science Council of Canada, Ottawa, 1976.

Chapter four is particularly relevant to undergraduate mathematics teaching and curriculum development. Their philosophy is very close to [CUPM 81]: increased emphasis on teaching the origin and methods of solution of problems in the mathematical sciences and the ability to communicate with other disciplines.

[CRICISAM 68] Calculus: a computer oriented approach, W. Steinberg and R.J. Walker, Center for Research in College Instruction of Science and Mathematics (CRICISAM), Florida State University, 1968.

An experimental textbook, that begins with approximating solutions of equations, convergence, finding areas and volumes by approximating and using limit theorems, leading into integration and differentiation with many numerical and algorithmic ideas. Read the reasons offered for its lack of success in [FUTURE 83], p.225.

[CUPM 81] Recommendations for a General Mathematical Sciences Program, Report of the Committee on the Undergraduate Program in Mathematics (CUPM), ed. A.C. Tucker, Mathematical Association of America, Washington, 1981.

A very well considered set of proposals about the undergraduate teaching of mathematics in general. Chapter one makes many valuable points and is worth reading in September every year.

[FUTURE 83] The Future of College Mathematics, ed. A. Ralston and G.S. Young, Springer-Verlag, New York, 1983.

A collection of invited papers for a Sloan Foundation conference in 1982 with brief reports on the discussion of these papers and three workshops. Most of the papers centre about the question of replacing introductory calculus by a discrete mathematics course, raised by [RALSTON 81]. Unfortunately most curriculum development is either too general in principle to see how it would be implemented or too specific in contents to catch a glimmer of the spirit. Course descriptions by H.S. Wilf (p.38) and F.S. Roberts (p.126) are the exceptions.

[GORDON 79a] "A discrete approach to computer-oriented calculus", S.P. Gordon, American Mathematical Monthly 86 (1979) 386-391.

Describes an approach to calculus which incorporates the computer "in a particularly natural way": finite differences and sums are used "to motivate the infinitesimal calculus and to provide the appropriate setting for solving "real" problems using discrete approximations". (It is claimed in [RALSTON 81] that the computer should influence the mathematics curriculum more profoundly than such local uses.) Contains a good bibliography of articles about incorporating computing in the calculus sequence.

[GORDON 79b] "A discrete approach to the calculus", S.P. Gordon, Int. J. Math. Educ. Sci. Technol. 10 (1979) 21-31.

Similar to [GORDON 79a], but with less mathematical contents and a more general discussion. Contains an analysis of experimental implementations of this approach.

[GREENSPAN 80] Arithmetic Applied Mathematics. D. Greenspan, Pergamon Press. 1980.

How finite techniques and the computer can replace much of what is done in continuous applied mathematics. From the preface: "In this book we will develop a computer, rather than a continuum, approach to the deterministic theories of praticle mechanics. (...) At those points where Newton, Leibniz, and Einstein found it necessary to apply the analytical power of the calculus, we shall, instead, apply the computational power of modern digital computers. (...) The price we pay for [the mathematical simplicity of our approach] is that we must do our arithmetic at high speeds." (See also the author's Discrete Models, Addison-Wesley, 1973.)

[HENRICI 77] Computational Analysis with the HP-25 Pocket Calculator, P. Henrici, Wiley, 1977.

Programs by the author implementing algorithms in number theory, equation solving, numerical integration, evaluation of special functions. Interesting in the way he uses many areas of mathematics to produce algorithms that are fast enough to run on a programmable pocket calculator.

[HILTON 79] The Role of Applications in the Undergraduate Mathematics Curriculum, Committee on Applied Mathematics Training (P.J. Hilton, chairman), Assembly of Mathematical and Physical Sciences, National Research Council, Washington, 1979.

Report of a committee formed by the NRC to consider the problem of reshaping the teaching of mathematics to meet the needs and purposes of today's students. Similar in spirit to [CUPM 81], with a greater stress on "the restoration of the role of differential equations in the core curriculum". Advocates closing the gap between abstract and applied mathematics and creating a broad major in the mathematical sciences. Identifies the principal problem as being one of attitude.

[HILTON 82] "The emphasis on applied mathematics today and its implications for the mathematics curriculum", P.J. Hilton, in: New Directions in Applied Mathematics, ed. P.J. Hilton and G.S. Young, Springer-Verlag, 1982, pp. 155-163.

By the chairman of the "Hilton panel" responsible for the report [HILTON 79]. As in the talk presented in [CMESG 83], promotes the unity of the mathematical sciences. "Might the sterile antagonism which one sometimes finds today between pure and applied mathematics-and pure and applied mathematicians-be eliminated by abandoning those labels and reverting to the notion of a single indivisible discipline, mathematics."

[KNUTH 74] "Computer science and its relation to mathematics", D.E. Knuth, American Mathematical Monthly 81 (1974) 323-343.

A personal view of the interactions between computer science and mathematics. Discussion of a "typical computer science problem"(hashing) to illustrate the similarities and differences between the two fields. Describes computer science as "the study of algorithms".

[LAX ET AL. 76] Calculus with Applications and Computing, Volume 1, P. Lax, S. Burstein and A. Lax, Springer-Verlag, 1976.

In [NOTICES 83], the first author supports the view that calculus should remain the centerpiece of mathematics education in the first two years of college. But it is essential, he adds, the modify the way it is taught according to the "modern spirit", for example by taking into account the impact of computing. His conceptions can be found in this inspiring text which emphasizes the relation of calculus to science. Numerical methods are presented as organic parts of calculus, not mere appendices, while change of variables or integration by parts serve to get new integrals easier to approximate numerically. A must read !

[McCARTY 75] Calculator Calculus, G. McCarty, Page-Ficklin, 1975.

A workbook to accompany a conventional calculus text. Uses the pocket calculator to illustrate the theory. Each chapter contains several examples with detailed discussions and complete solutions, easy exercices and more difficult problems. Most important theorems are usually cited explicitly.

[MIEL 80] "Calculator calculus and roundoff errors", G. Miel, American Mathematical Monthly 87 (1980) 243-253.

Although the calculator can reinforce understanding of calculus notions, indiscriminate use of it or lack of awareness of the effects of roundoff errors can lead to mistaken interpretations of results. Nicely illustrates the point with many examples, for instance Zeno's paradox seen in the context of roundoff errors. Extensive bibliography.

[MILES 71] "Calculus and the computer, the CRICISAM calculus project - Past, present and portent", E.P. Miles, Jr., American Mathematical Monthly 78 (1971) 284-291.

On the history of the CRICISAM calculus project. (Informations on CRICISAM can also be found in Miles' paper in the book [TOMORROW 81].)

[MODELS 73] Mathematical Models and Applications. D.P. Maki and M. Thompson, Prentice-Hall, 1973.

Intended for junior and senior students. Topics covered include Markov chains, linear optimization, graphs and growth processes (both by means of diffential and finite difference equations). Each chapter has exercises, practic projects and a good bibliography. Authors suggest a variety of courses that can be taught from the book: survey, in-de teacher preparation.

[MODELS 77] Mathematical Models, R. Haberman, Prentice-Hall, 1977.

Intended for the junior-senior level. Three long chapters (over 100 pages each) on mechanical vibrations, population dynamics and traffic flow. Pleasant to read, a very good text for an undergraduate introduction to techniques of applied mathematics. Chapter on population dynamics uses both a discrete and a continuous approach.

[NOTICES 83] "Freshman Mathematics" Notices of the American Mathematical Society 30, 1983, pp. 166-171. Transcript of a panel discussion on this topic at the Annual Meeting in Denver in 1983, with speakers A. Ralston, P.D. Lax, G.S. Young, and R.O. Wells Jr. Noteworthy is Ralston's remark that "we need an alternative to freshman calculus which is not a revolution but an evolution," and Lax's rebutal to the discrete mathematics suggestions.

[RALSTON 81] "Computer science, mathematics and the undergraduate curricula in both," A. Ralston, American Mathematical Monthly 88, 1981, pp. 472-485.

The article that provoked the conference [FUTURE 83], arguing for the consideration of a separate mathematics curriculum for computer science undergraduates, beginning with a discrete mathematics course rather than calculus. A well-argued introduction to the topic which notes that, however desirable and valuable, the use of computers in the calculus and other courses is not sufficient. "There has been little realization that the advent of computers and computer science might suggest some fundamentale changes in the [mathematics] undergraduate curriculum." (See also the longer technical report, with the same title, on which this paper is based: Technical report 161, Dept. of Computer Science, SUNY at Buffalo, 1980.)

[ROSSER 72] "Mathematics courses in 1984", J.B. Rosser, Amer. Math. Monthly 79 (1972) 635-648.

"Unless we revise the calculus course, and the differential equations course, and probably the linear algebra course, and I do not know what other courses, so as to embody much use of computers, most of the clientele for these courses will instead be taking computer courses in 1984."

"I believe [the average student] would be letter offlearning the algorithms in a mathematical course, if they are property taught there. If they are property taught!"

[STRANG 80] Linear Algebra and its Applications (Second Edition), G. Strang, Academic Press, New York, 1980. An exciting read in applied linear algebra, with computational discussions including the stability and efficiency of the algorithms, and notes on LINPACK. Well-designed from the view point of numerical linear algebra, but weak on the concept and use of linear transformations.

[SUPPLEMENT 79] Pocket Calculator Supplement for Calculus, J.B. Rosser and C. de Boor, Addison Wesley, New York, 1979. Inexpensive soft cover edition, can be used in conjunction with any calculus text.

[TOMORROW 81] Mathematics Tomorrow, ed. L.A. Steen, Springer-Verlag, 1981.

A collection of essays, some of which are related to mathematics curriculum, applications and the influence of computers. For example, the essays by J.P. King ("But I ask a favor, let one course, just one, remain pure. And let it be beginning calculus."), A. Tucker (same spirit as Chapter I of [CUPM 81]) and W.F. Lucas ("[The mathematics community] must act quickly and in a meaningful way. There exists many good strategies. The question is whether we will select one and implement it in time, or whether we will follow philosophy's decline into prestigious isolation and irrelevance.").

[WILF 82] "The disk with the college education", H.S. Wilf, American Mathematical Monthly 89 (1983) 4-8.

muMATH is comming! muMATH is comming! (The impact of the availability of symbolic manipulation on personal computers is also discussed in the author's paper in [FUTURE 83], where it is stressed that "it can be very unsettling to realize that what we previously thought was a very human ability (...) can actually be better done by 'machines".")

[WONNACOTT 77] Calculus: an Applied Approach, T.H. Wonnacott, Wiley, New York, 1977.
Begins with approximating and defining e and has much interplay between the differencing situations ("finite calculus") and differentiation situations ("infinitesimal calculus"). His teaching ability shows in this readable text.

TOPIC GROUPS

L INTUITIONS AND FALLACIES IN REASONING ABOUT PROBABLITY - BY DANIEL KAHNEMAN
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M MATHEMATICS CURRICULUM DEVELOPMENT IN CANADA: A PROJECTION FOR THE FUTURE
- BY TOM KIEREN
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TOPIC GROUP L

PUBLISHED MATERIAL RELATED TO KAHNEMAN AND ASSOCIATES RESEARCH, UPON WHICH THIS PRESENTATION WAS BASED, IS INCLUDED IN:

JUDGEMENTS UNDER UNCERTAINTY: HEURISTICS AND BIASES, BY KAHNEMAN, SLORIC AND TVERSKY, CAMBRIDGE UNIVERSITY PRESS, 1982.

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